Summary: – Sect 4.5
– Sect 4.6.

4.5 Dimension of a vector space

Recall $B = \{b_1, b_2, \dots, b_n\}$ is a basis for V if:

(i) $\{b_1, b_2, \dots, b_n\}$ is linearly independent
(ii) Span $\{b_1, b_2, \dots, b_n\} = V$

Theorem (Unique representation)

If $B = \{b_1, b_2, \dots, b_n\}$ is a basis for the vector space V then for any x in V there exists a unique set of scalars $c_1, c_2, \dots, c_n$ such that

$$x = c_1 b_1 + c_2 b_2 + \dots + c_n b_n \qquad (1)$$

Proof Since span $\{b_1, b_2, \dots, b_n\} = V$ a set of scalars satisfying (1) must exist.

Suppose there are two such sets:

$$x = c_1 b_1 + c_2 b_2 + \dots + c_n b_n$$
$$x = d_1 b_1 + d_2 b_2 + \dots + d_n b_n$$

Then

$$
\begin{aligned}
0 = x + (-1)x &= c_1 b_1 + c_2 b_2 + \dots + c_n b_n + (-d_1) b_1 + (-d_2) b_2 \\
&\quad + \dots + (-d_n) b_n \\
&= (c_1 - d_1) b_1 + (c_2 - d_2) b_2 + \dots + (c_n - d_n) b_n
\end{aligned}
$$

Since $\{b_1, b_2, \dots, b_n\}$ are linearly independent we must have $c_1 - d_1 = 0$, $c_2 - d_2 = 0$, $\dots$, $c_n - d_n = 0$ or $c_1 = d_1$, $c_2 = d_2$, $\dots$, $c_n = d_n$. So the representation (1) is unique!

<u>Example</u> $\{1, t, t^2\}$ is a basis in $\mathbb{P}_3$

$p(t) = 3 + 4t - t^2$ is in $\mathbb{P}_3$. The representation of $p(t)$ in terms of the basis $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$ is

$$p(t) = 3p_1 + 4p_2 + (-1)p_3$$

<u>Theorem 9</u> If a vector space V has a basis $B = \{b_1, b_2, \dots, b_n\}$ then any set in V that contains more than n vectors is linearly dependent

Proof. Let $\{v_1, v_2, \dots, v_p\}$ be a set in V with more than n vectors, $p > n$. Then by the unique representation theorem there exists the vectors in $\mathbb{R}^n$:

$$\hat{v}_1 = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix}, \hat{v}_2 = \begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{bmatrix}, \dots, \hat{v}_p = \begin{bmatrix} c_{1p} \\ c_{2p} \\ \vdots \\ c_{np} \end{bmatrix}$$

(2) Such that $v_j = c_{1j} b_1 + c_{2j} b_2 + \dots + c_{nj} b_j$

But $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_p\}$ is linearly dependent because there are more vectors than the number of entries in each vector ($p > n$). So there exist $\alpha_1, \alpha_2, \dots, \alpha_p$ in $\mathbb{R}$ not all zero such that

$$\alpha_1 \hat{v}_1 + \alpha_2 \hat{v}_2 + \dots + \alpha_p \hat{v}_p = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now, from (2) we have.

$$\begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p &= (\alpha_1 c_{11} + \alpha_2 c_{12} + \dots + \alpha_p c_{1p}) b_1 \\ &\quad + (\alpha_1 c_{21} + \alpha_2 c_{22} + \dots + \alpha_p c_{2p}) b_2 + \dots \\ &= 0 \cdot b_1 + 0 \cdot b_2 + \dots + 0 \cdot b_p = 0 \end{aligned}$$

Hence $v_1, v_2, \dots, v_p$ are linearly dependent.

Corollary If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Def (i) If V is spanned by a finite set then V is called finite dimensional. The dimension of V:

$$\dim V = \text{the number of vectors in a basis of } V$$

(ii) $\dim \{0\} = 0$ by definition

(iii) If V is not spanned by a finite set, the V is called infinite dimensional

Examples $\dim \mathbb{R}^n = n$, $\dim \mathbb{P}_2 = 3$, $\dim \mathbb{P}_n = n+1$

$\mathbb{P}$ = the vector space of all polynomials is infinite dimensional.

Remark 1 If A is a matrix then
$$\dim \operatorname{col} A = \text{number of pivot columns in } A$$

Remark 2 If A is a matrix then

$$\dim \operatorname{Nul} A = \text{number of non pivot columns in } A$$

4.6. The rank of a matrix

Def If A is a matrix then
$$\operatorname{rank} A = \dim \operatorname{col} A$$

Theorem If A is an $m \times n$ matrix then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

Proof This is a simple consequence of:

$$\begin{matrix}\text{number of} \\ \text{pivot columns}\end{matrix} + \begin{matrix}\text{number of} \\ \text{non pivot columns}\end{matrix} = \begin{matrix}\text{number of} \\ \text{columns}\end{matrix}$$

Theorem (Rank and invertible matrix theorem)
Let A be an $n \times n$ square matrix. The following are equivalent:

a) A is invertible
b) $\operatorname{Rank} A = n$
c) $\dim \operatorname{Nul} A = 0$ (which means $\operatorname{Nul} A = \{0\}$)