Lecture 2 Math 225

Note Title 1/17/2008

Summary

Section 1.1 (continuation) Matrices and row equivalent matrices

Section 1.2 Echelon matrices and reduced echelon matrices.

Section 1.1 (continuation)

<u>Other examples</u>

Similar to Example 2.1

matrix of the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 4x_2 + 8x_3 = 4 \\ 3x_1 + 5x_2 + 11x_3 = 6 \end{cases} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 5 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 4 \\ 3 & 5 & 11 & 6 \end{bmatrix} \quad \begin{matrix} \text{row } 2 \mapsto \text{row } 2 - 2\,\text{row } 1 \\ \sim \\ \text{row } 3 \mapsto \text{row } 3 - 3\,\text{row } 1 \end{matrix} \quad \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & -1 & 2 & 3 \end{bmatrix}$$

extended matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & -1 & 2 & 3 \end{bmatrix} \underset{\text{swap row 2 and 3}}{\sim} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix} *$$

this is the echelon form of the matrix, see section 1.2 for a formal definition. It is now time to solve for the last variable x_3 and eliminate it from the other equations:

$$* \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix} \underset{\text{divide row 3 by 2}}{\sim} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} *$$

$$* \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underset{\text{row } 1 \mapsto \text{row } 1 - 3\,\text{row } 3}{\overset{\text{row } 2 \mapsto \text{row } 2 - 2\,\text{row } 3}{\sim}} \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} *$$

$$* \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underset{\text{row 2 by } -1}{\overset{\text{multiply}}{\sim}} \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} *$$

$$* \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{\text{row } 1 \mapsto \text{row } 1 - 2\,\text{row } 2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} **$$

Which is equivalent to

$$\begin{cases} x_1 & & & = 0 \\ & x_2 & & = -1 \\ & & x_3 & = 1 \end{cases}$$

Similar to example 1.2:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 5x_2 + 8x_3 = 4 \end{cases} \qquad \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 8 & 4 \end{bmatrix} \overset{\text{row } 2 \mapsto \text{row } 2 - 2\,\text{row } 1}{\sim} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix} *$$

$$* \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \overset{\text{row } 1 \mapsto \text{row } 1 - 2\,\text{row } 2}{\sim} ** \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

This is the simplest form in which we can write this matrix (and system), see the theory below. Hence

$$\begin{cases} x_1 & - x_3 = -3 \\ x_2 & + 2x_3 = 2 \end{cases} \iff \begin{cases} x_1 = -3 + x_3 \\ x_2 = 2 - 2x_3 \\ x_3 = \text{free} \end{cases}$$

Remark this last system has infinitely many solutions

Question How do we know we reached the simplest form? The answer follows

Def The leading entry in a row is the leftmost nonzero element in that row

Def A matrix is in echelon form (or row echelon form) if:

1. any nonzero row is above any zero row;
2. the leading entry of any row is in a column to the right of the leading entry of the row above it;
3. all entries in a column below a leading entry are zeros.

A matrix is in reduced echelon form if besides satisfying 1, 2, 3 above it also verifies:

4. all leading entries are 1;
5. each leading entry is the only nonzero element on its column

Remark In the above examples matrices marked by "*" are in echelon form while the ones marked by "**" are in reduced echelon form.

Theorem Any matrix is row equivalent to one and only one reduced echelon matrix

Conclusion Reduced echelon form is the simplest form in which a matrix can be brought via row equivalent operations. The algorithm should stop when this form has been reached.