Lecture 20 Math 225

Note Title 4/9/2008

Summary:

5.1. Eigenvectors and Eigenvalues
5.2 The characteristic equation

5.1. Eigenvectors and Eigenvalues

Consider an $n \times n$ square matrix A. There are special vectors in $\mathbb{R}^n$ and special scalars (numbers) associated to each such matrix:

Def (eigenvector) x in $\mathbb{R}^n$ is an eigenvector for A if $x \neq 0$ and

(1) $$Ax = \lambda x \text{ for some } \lambda \text{ in } \mathbb{R}$$

(eigenvalue) λ in $\mathbb{R}$ is an eigenvalue for A if the equation

(2) $$Ax = \lambda x$$

has at least one nontrivial solution $x \neq 0$.

Example 1 $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ an eigenvector for A? How about $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

Answer

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ Yes $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

is an eigenvector for A. The corresponding eigenvalue is 3!

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$ for any

λ in $\mathbb{R}$. So $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is not an eigenvector for A.

Example 2 $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Is 2

an eigenvalue for A? If yes find its corresponding eigenvectors.

$$Ax = 2x$$

$$\Leftrightarrow \quad (A - 2I_d)x = 0 \qquad (3)$$

The latter has a nonzero solution if and only if

$$\det(A - 2I_d) = 0$$

$$\begin{vmatrix} 4-2 & -1 & 6 \\ 2 & 1-2 & 6 \\ 2 & -1 & 8-2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{vmatrix} = 0$$

Since the first two rows are the same!

Yes 2 is an eigenvalue for A.

To find the corresponding eigenvectors one needs to solve equation (3) above:

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad \begin{array}{l} x_1 = 1/2\, x_2 - 3x_3 \\ x_2, x_3 \text{ are free} \end{array}$$

In vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

So the eigenvectors corresponding to eigenvalue 2 are all the vectors in the plane spanned by $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

<u>Remark</u> If λ is an eigenvalue of the $n \times n$ matrix A then the corresponding eigenvectors form the subspace:

$$\mathrm{Nul}\,(A - \lambda I_d)$$

of $\mathbb{R}^n$ called the eigenspace of λ. A basis for this space can be found by finding a basis for $\mathrm{Nul}\,(A - \lambda I_d)$, see the previous example and Section 4.3.

5.2 The characteristic equation

How does one find the eigenvalues and the eigenvectors of a given $n \times n$ square matrix A?

Step 1 Find the eigenvalues:

$$Ax = \lambda x \text{ must have solutions } x \neq 0$$

$$\Leftrightarrow (A - \lambda I_d)x = 0 \text{ must have solutions } x \neq 0$$

$$\Leftrightarrow \det(A - \lambda I_d) = 0$$

The characteristic equation

Step 2 For each λ solving the characteristic equation one finds the corresponding eigenvectors by determining the subspace:

$$Nul(A - \lambda I_d)$$

or, equivalently by finding all solutions of $(A - \lambda I_d)x = 0$

__Example 3__ Find all eigenvalues and eigenvectors of :

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

__Step 1__

$$\det(A - \lambda I_{d}) = 0$$

$$\Leftrightarrow \quad \begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \quad (2-\lambda)(4-\lambda) - 1 = 0$$

$$\Leftrightarrow \quad \lambda^2 - 6\lambda + 7 = 0$$

$$\Leftrightarrow \quad \lambda_1 = \frac{6 - \sqrt{36-28}}{2} = 3 - \sqrt{2}$$

$$\lambda_2 = \frac{6 + \sqrt{36-28}}{3} = 3 + \sqrt{2}$$

<u>Step 2</u> Eigenvectors corresponding to $\lambda_1 = 3-\sqrt{2}$

$$\begin{bmatrix} 2-\lambda_1 & 1 \\ 1 & 4-\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \sqrt{2}-1 & 1 & 0 \\ 1 & \sqrt{2}+1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{2}+1 & 0 \\ \sqrt{2}-1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \sqrt{2}+1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix} \quad x_2 \text{ in } \mathbb{R}.$$

Eigenvectors corresponding to $\lambda_2 = 3+\sqrt{2}$

$$\begin{bmatrix} 2-\lambda_2 & 1 \\ 1 & 4-\lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1-\sqrt{2} & 1 & 0 \\ 1 & 1-\sqrt{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1-\sqrt{2} & 0 \\ -1-\sqrt{2} & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1-\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \sqrt{2}-1 \\ 1 \end{bmatrix}$$

Shortcut: If A is upper or lower triangular then the eigenvalues of A are the entries on its main diagonal.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn} \end{bmatrix} - \lambda I_d =$$

$$= \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn}-\lambda \end{vmatrix} =$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

$$\Rightarrow \lambda_1 = a_{11},\ \lambda_2 = a_{22},\ \ldots,\ \lambda_n = a_{nn}$$