5.3. Diagonalization of square matrices and Applications.

Def (similarity) $n \times n$ matrices A and B are similar if there exists an invertible matrix P such that

$$B = P^{-1} A P$$

Remark 1 If $B = P^{-1} A P$ then

$$B^k = \underbrace{(P^{-1}AP)(P^{-1}AP)\dots(P^{-1}AP)}^{k \text{ factors}}$$

$$= P^{-1} A(PP^{-1})A(PP^{-1})\dots A(PP^{-1})A P$$

$$= P^{-1} A^k P$$

So if B^k is easy to compute then

$A^k = P B^k P^{-1}$ is also easy to compute.

Def (diagonalizable) A $n \times n$ matrix A is diagonalizable if it is similar to a diagonal matrix (a matrix that has all the entries zero except on the diagonal).

Remark 2 If A is similar to

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

then $D = P^{-1}AP$ for some invertible matrix P and

$$A^k = PD^kP^{-1} = P \begin{bmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix} P^{-1}$$

Theorem An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

In this case

$$\begin{bmatrix} a_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = P^{-1}AP$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A and $P = [v_1 \ v_2 \ \dots \ v_n]$ is the matrix formed by the corresponding linearly independent eigenvectors.

<u>Examples</u> In Lecture 20 we saw that

(1) $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ has eigenvalues

$\lambda_1 = 3-\sqrt{2}$ with eigenvector $v_1 = \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix}$

$\lambda_2 = 3+\sqrt{2}$ with eigenvector $v_2 = \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$

v_1 and v_2 are linearly independent (check!)
Therefore A is diagonalizable and

$$\begin{bmatrix} 3-\sqrt{2} & 0 \\ 0 & 3+\sqrt{2} \end{bmatrix} = P^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} P$$

where $P = \begin{bmatrix} -1-\sqrt{2} & -1+\sqrt{2} \\ 1 & 1 \end{bmatrix}$

$$P^{-1} = \frac{1}{-2\sqrt{2}} \begin{bmatrix} 1 & 1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{4} & \frac{1}{2} - \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{1}{2} + \frac{\sqrt{2}}{4} \end{bmatrix}$$

(2) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

eigenvalues are $\lambda_1 = \lambda_2 = 2$ (matrix is upper triangular)

$$\mathrm{Nul}\,[A - 2\,I_d] = \mathrm{Nul} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

So A has only one linearly independent eigenvector
Therefore A is NOT DIAGONALIZABLE !

(3) $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Characteristic equation:

$$\det(A - \lambda I_d) = 0 \iff -(\lambda - 1)(\lambda + 2)^2 = 0$$

$\iff$ eigenvalues $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -2$.

$$\mathrm{Nul}\,(A - I_d) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\mathrm{Nul}\,(A + 2I_d) = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A has three linearly independent eigenvectors.
Therefore A is diagonalizable and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = P^{-1} A P$$

where $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Conclusion: To check whether a given $n \times n$ matrix A is diagonalizable do the following:

Step 1 Find all its eigenvalues by solving

$$\det(A - \lambda I_d) = 0$$

Step 2 Find a basis for

$$\text{Nul}(A - \lambda I_d) \text{ for each eigenvalue } \lambda$$

An important result (see Theorem 2 page 307 in the textbook) says that the vectors found at this step are linearly independent.

Step 3 Count the vectors found in Step 2. If they are n then A is diagonalizable. If they are less than n A is NOT DIAGONALIZABLE

Remark If an $n \times n$ matrix has n distinct eigenvalues then it is diagonalizable

Application

$$x_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix} = \begin{bmatrix} \text{number of owls in month } k \\ \text{number (in thousands) of rats in month } k \end{bmatrix}$$

It is known that:

$$x_{k+1} = A x_k \quad \text{where } A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$$

in other words:

$$O_{k+1} = 0.5\, O_k + 0.4\, R_k$$
$$R_{k+1} = -.104\, O_k + 1.1\, R_k$$

So half the owls survive one month without rats (food) and the rats grow by 10% in one month without owls.

What happens in the long run?

<u>Answer</u> $x_k = A^k x_0$

A is diagonalizable with eigenvalues:

$\lambda_1 = 1.02$ eigenvector $v_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$

$\lambda_2 = .58$ eigenvector $v_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

By Remark 2:

$$x_k = P \begin{bmatrix} (1.02)^k & 0 \\ 0 & (0.58)^k \end{bmatrix} P^{-1} x_0 \qquad (*)$$

where

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 13 & 1 \end{bmatrix}$$

Since $\{v_1, v_2\}$ is a basis in $\mathbb{R}^2$ we can write:

$$x_0 = c_1 v_1 + c_2 v_2$$

Then $(*)$ becomes:

$$x_k = (1.02)^k c_1 v_1 + \underbrace{(0.58)^k c_2 v_2}_{\text{converges to } 0.}$$

For k large:

$$\begin{bmatrix} O_k \\ R_k \end{bmatrix} = x_k \approx (1.02)^k c_1 \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

in other words both populations grow by 2% a month and there are 10 owls at every 13 (thousands) rats.