Lecture 22 Math 225

Note Title 4/25/2008

6.1 Inner product, length, distance, orthogonality.

Def (Inner product) If U, V are vectors in $\mathbb{R}^n$:

$$U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}, V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

then their inner product (dot product) is the number:

$$U \cdot V = U_1 V_1 + U_2 V_2 + \dots + U_n V_n$$

Examples 1° $U = [1, 2, 3]^T$ $V = [4, 5, 6]^T$

$$U \cdot V = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

2° $$\mathbf{0} \cdot U = 0 \cdot U_1 + 0 \cdot U_2 + \dots + 0 \cdot U_n = 0$$

Remark $u \cdot v = \underbrace{u^T v}_{\text{product of two matrices}}$

Properties of the inner product:

1° $u \cdot u \geq 0$ $u \cdot u = 0$ if and only if $u = 0$
2° $u \cdot v = v \cdot u$
3° $(u+v) \cdot w = u \cdot w + v \cdot w$
4° $c(u \cdot v) = (cu) \cdot v = u \cdot (cv)$ for c in $\mathbb{R}$.

Def (length of a vector) If u is a vector in $\mathbb{R}^n$ the length of u is the number:

$$\|u\| = \sqrt{u \cdot u}$$

Remarks (i) If $u = [u_1, u_2, \dots, u_n]^T$ then

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

(ii) $\|u\| = 0$ if and only if $u = 0$

Example $\|[1, 2, 3]^T\| = \sqrt{14}$

<u>Def</u> (distance between two vectors) If u, v are vectors in $\mathbb{R}^n$ then the distance between them is the number.

$$\text{dist}(u, v) = \|u - v\|$$

<u>Remark</u> If $u = [u_1, u_2, \dots, u_n]^T$ and $v = [v_1, v_2, \dots, v_n]^T$ then

$$\text{dist}(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2}$$

Hence this notion generalizes the distance between points in $\mathbb{R}^2$ and $\mathbb{R}^3$.

<u>Example</u>:

$$\text{dist}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right) = \sqrt{(-3)^2 + (-3)^2 + (-3)^2} = \sqrt{27}$$

which is exactly the length of the segment with endpoints at coordinates $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

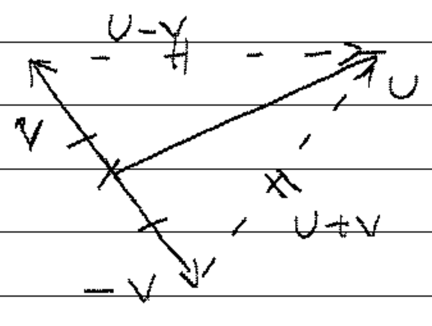
<u>Def</u> (orthogonal vectors) If u, v are vectors in $\mathbb{R}^n$ they are called orthogonal (perpendicular) if
$$u \cdot v = 0$$

<u>Examples</u> $[1,2,3]^T$ and $[4,5,6]^T$ are not orthogonal because
$$[1,2,3]^T \cdot [4,5,6]^T = 32 \neq 0$$

$[3,1,1]^T$ and $[-1,2,1]^T$ are orthogonal because:
$$[3,1,1]^T \cdot [-1,2,1]^T = -3+2+1 = 0$$

<u>Remark</u> The orthogonality of vectors generalizes the notion of perpendicular lines in plane or space. Indeed the following argument shows that:

$u \cdot v = 0$ if and only if the lines $\mathrm{Span}\{u\} \perp \mathrm{Span}\{v\}$ (the angle between the two lines is $90°$).



$\mathrm{Span}\{u\} \perp \mathrm{Span}\{v\}$ if and only if
$$\|u+v\| = \|u-v\|$$

But

(*) $$\|U+V\|^2 = (U+V)\cdot(U+V) = U\cdot(U+V) + V\cdot(U+V)$$
$$= U\cdot U + U\cdot V + V\cdot U + V\cdot V$$
$$= \|U\|^2 + \|V\|^2 + 2U\cdot V$$
$$\|U-V\|^2 = (U-V)\cdot(U-V) = U\cdot(U-V) - V\cdot(U-V)$$
$$= U\cdot U - U\cdot V - V\cdot U + V\cdot V$$
$$= \|U\|^2 + \|V\|^2 - 2U\cdot V$$

So

$\|U+V\| = \|U-V\|$ iff $\|U+V\|^2 = \|U-V\|^2$ iff
$2U\cdot V = -2U\cdot V$ iff $U\cdot V = 0$

<u>Theorem</u> (Pythagoras) U, V are orthogonal if and only if:

$$\|U+V\|^2 = \|U\|^2 + \|V\|^2$$

<u>Proof</u> From (*) above

$$\|U+V\|^2 = \|U\|^2 + \|V\|^2 + 2U\cdot V$$

So $\|U+V\|^2 = \|U\|^2 + \|V\|^2$ iff $U\cdot V = 0$.