Lecture 3 Math 225

Note Title 1/21/2008

Summary

Section 1.2 (cont) Pivot positions and columns

Section 1.3 Vector Equations

Section 1.2 Pivot positions and columns

Recall : System of linear equations
$\longmapsto$ extended matrix $\longmapsto$ echelon form
$\longmapsto$ reduced echelon form $\longmapsto$ solutions

Example

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 0\cdot x_1 + x_2 + x_3 = 2 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0 \end{cases} \longmapsto \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is echelon form. For reduced echelon form eliminate the 2 above the leading entry in row 2:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtained the reduced echelon form so we STOP.

<u>Def</u> A pivot position in a matrix is the position of a leading one in its reduced echelon form.

A pivot column is a column on which there is a pivot position.

<u>Remark</u>: In the example above column 1 row 1 and column 2 row 2 are pivot positions. Columns 1 and 2 are pivot columns.

<u>Note</u>: Nonzero leading entries in the echelon form also give the pivot positions.

<u>Interpretation</u>: Pivot columns correspond to basic (or determined) variables while the other columns correspond to free variables

In the example above: x_1 and x_2 are basic variables while x_3 is free. So the general solution is:

$$\begin{cases} x_1 = -3 + x_3 \\ x_2 = 2 - x_3 \\ x_3 = \text{free} \end{cases}$$

<u>Another example</u>:

$$\begin{cases} 0 \cdot x_1 + x_2 + 2x_3 + 2x_4 + x_5 = 1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + x_4 + x_5 = 2 \\ 0 \cdot x_1 - 3x_2 - 6x_3 - 5x_4 - 2x_5 = 0 \end{cases}$$

pivot position

neglect

$$\begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & -3 & -6 & -5 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

pivot column

pivot column

$$\sim \begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot column

The system has no solutions (inconsistent) because the last equation has become:

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 = \underline{1}$$

Interpretation If the extended matrix of a system has the last column as a pivot column then the system has no solutions (is inconsistent)

Section 1.3 Vectors and operation with vectors.

Def (algebraic definition of vectors) An n-vector is an n-tuple of real numbers. Since the n-tuple can be arranged in a column we can identify vectors with matrices with one column

Examples : $\begin{bmatrix} 1.25 \\ -2.1 \end{bmatrix}$ is a 2-vector

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a 3-vector $\begin{bmatrix} -1 \\ 0 \\ 2 \\ -4 \end{bmatrix}$ is a 4-vector

Equality of two vectors:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$ if and only if $u_1 = v_1, u_2 = v_2, \ldots$ $u_n = v_n$

Examples $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Addition of two vectors

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$ the result is called the sum

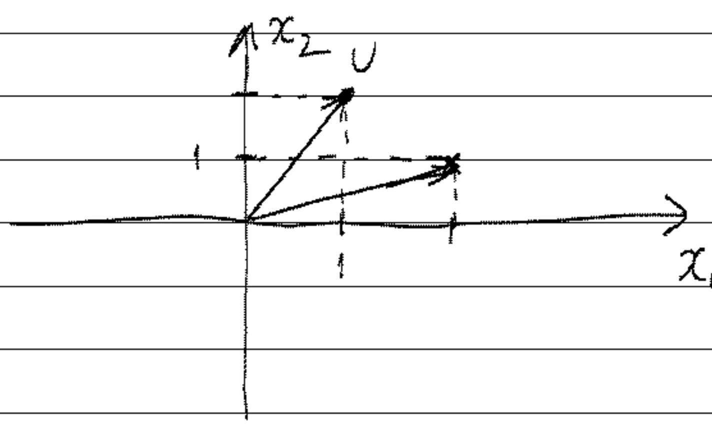
Examples $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$

Multiplication with scalars

$$c \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$

Examples : $(-2) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ $\qquad 3 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$

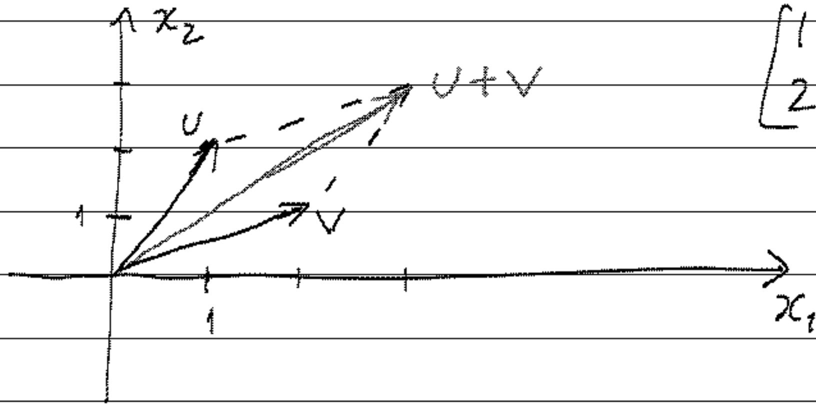
Remark : A 2-vector can be identified with a point in plane once a rectangular coordinate system has been chosen :



$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

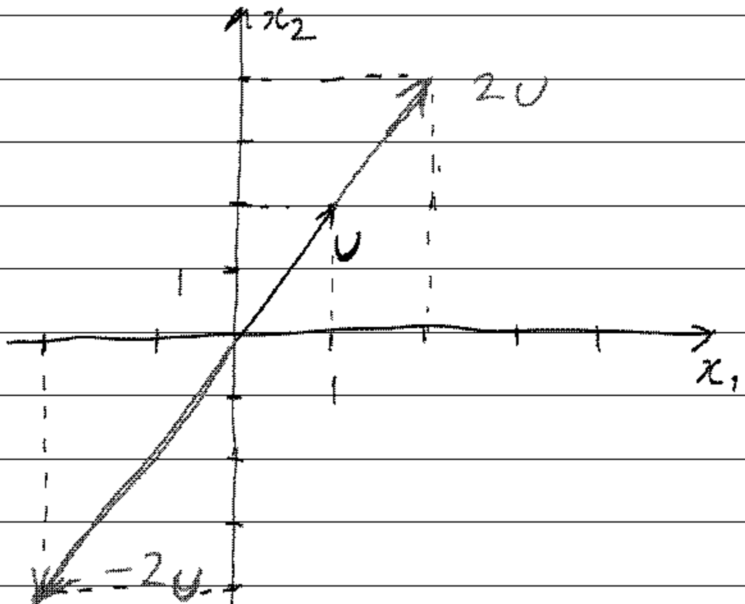
$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Addition of vectors becomes the parallelogram rule :



$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$$

Multiplication with scalars is equivalent with dilations (or contractions) of the length of the vector and, if the scalar is negative, to a change to the opposite direction.



$$2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-2\\-4\end{bmatrix}$$

Similar for 3-vectors in space.

<u>Properties of addition and multiplication with scalars</u>:

(i) $u + v = v + u$
(ii) $(u+v)+w = u+(v+w)$
(iii) $u + 0 = 0 + u = u$, for $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

(iv) $u + (-u) = (-u) + u = 0$ for $-u = (-1)u$

(v) $c(u+v) = cu + cv$
(vi) $(c+d)u = cu + du$
(vii) $c(du) = (cd)u$
(viii) $1u = u$

<u>Remark</u> All the properties follow directly from the definitions of addition and multiplication with scalars and the properties of addition an multiplication of real numbers.