Lecture 4 Math 225

Note Title 1/24/2008

Summary

Sect 1.3 (cont) Linear combination of vectors, span, vector equations

<u>Def</u> $\mathbb{R}^n$ collection of all n-vectors (n-tuples) with real entries.

<u>Note</u> Addition of vectors is only defined for vectors with the same number of entries

<u>Def</u> (linear combination) If $v_1, v_2, \dots, v_p$ are in $\mathbb{R}^n$ and $c_1, c_2, \dots, c_p$ are scalars (real numbers) then

$$y = c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

is the linear combination of $v_1, v_2, \dots, v_p$ with weights $c_1, c_2, \dots c_p$

<u>Examples</u> $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

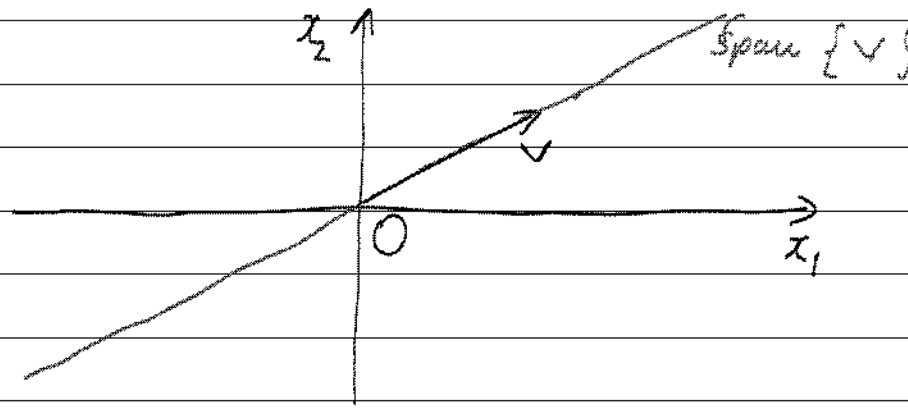
$$0v_1 + 0v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-1)v_1 + 2v_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

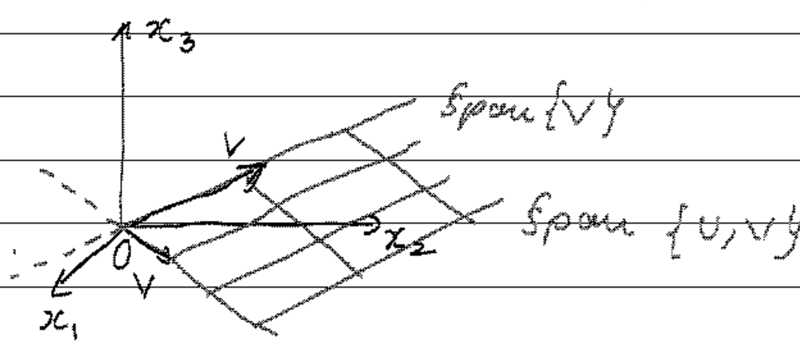
Def (Span) If $v_1, v_2, \ldots, v_p$ are in $\mathbb{R}^n$ then $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the set of all linear combinations of $v_1, v_2, \ldots, v_p$.

Examples in $\mathbb{R}^2$ if $v \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then

$\text{Span}\{v\}$ is a line in the plane through O



in $\mathbb{R}^3$ if $v \neq 0$ then $\text{Span}\{v\}$ is a line through O
if u is not a scalar multiple of v
then $\text{Span}\{u, v\}$ is a plane through O



Question that often arises in practice:

Given vectors $v_1, v_2, \dots, v_p$ and y in $\mathbb{R}^n$ can y be written as a linear combination of $v_1, \dots, v_p$? Equivalently, is y in Span$\{v_1, \dots, v_p\}$?

To answer it you must find $x_1, x_2, \dots, x_p$ such that

$$\underbrace{x_1 v_1 + x_2 v_2 + \dots + x_p v_p = y}_{\text{This is a vectorial eq.}}$$

<u>Example</u> Is $\underbrace{\begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}}_{y}$ in Span $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}}_{v_2} \right\}$?

Answer: Solve for x_1, x_2:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}$$

Same as $$\begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ 2x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}$$

Same as

$$\begin{bmatrix} x_1 - x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}$$

Same as

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 + 2x_2 &= 7 \\ x_1 + 3x_2 &= 9 \end{aligned}$$

We need to row reduce:

$$\begin{array}{c} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{bmatrix} \\ \begin{array}{ccc} \;v_1 & \quad v_2 & \quad y \end{array} \end{array}$$

The result is yes $\underset{x_1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \underset{x_2}{2} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}$

(check by solving the system)

In general : a vector equation :

$$x_1 V_1 + x_2 V_2 + \dots + x_p V_p = y$$

has the same solution as the system of linear equations with augmented matrix :

$$\begin{bmatrix} V_1 & V_2 & \dots & V_p & y \end{bmatrix}$$