Summary

1.4. The equation $Ax = b$

Previously: A linear combination of $a_1, a_2, \dots, a_n$ in $\mathbb{R}^m$ with weights $x_1, x_2, \dots, x_n$ is:

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

New notation:

Def For $A = [a_1 \ a_2 \ \dots \ a_n]$ an $m \times n$ matrix and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ a vector in $\mathbb{R}^n$ we denote by

Ax the vector:

$$Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

Example:

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}$$

<u>Properties of multiplication of a matrix and a vector</u>

If A is a $m \times n$ matrix u, v are vectors in $\mathbb{R}^n$ and c is in $\mathbb{R}$ then:

1° $A(u+v) = Au + Av$

2° $A(cu) = c(Au)$

<u>Proof</u>: Let $A = [a_1, a_2, \ldots, a_n]$ with $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^m$ and

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$ Then:

by definition of multiplication of a matrix and a vector

$$A(u+v) = (u_1+v_1)\,a_1 + (u_2+v_2)\,a_2 + \dots + (u_n+v_n)\,a_n$$

$$= u_1 a_1 + v_1 a_1 + u_2 a_2 + v_2 a_2 + \dots + u_n a_n + v_n a_n$$

Property (vi)
of multiplication
of vectors with
scalars

$$= (u_1 a_1 + u_2 a_2 + \dots + u_n a_n) + (v_1 a_1 + v_2 a_2 + \dots + v_n a_n)$$

Properties (i) and (ii)
of addition of vectors

$$= Au + Av$$

by definition of multiplication of a matrix and a vector

For 2°: $A(cu) = (cu_1)\,a_1 + (cu_2)\,a_2 + \dots + (cu_n)\,a_n$

$$= c(u_1 a_1) + c(u_2 a_2) + \dots + c(u_n a_n)$$

(vii)

$$= c(u_1 a_1 + u_2 a_2 + \dots + u_n a_n)$$

(v)

$$= c(Au)$$

Another way to calculate Ax

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{matrix} \nearrow & \nearrow & \nearrow \\ a_1 & a_2 & a_n \end{matrix}$$

By definition:

$$Ax = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \end{bmatrix}$$

To obtain the i-th component of Ax add the products of the entries in the i-th row of A with corresponding entries in x.

For A and x defined on previous page and

$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ the equation:

$$(1) \qquad A x = b$$

is equivalent to the vector equation:

$$(2) \qquad x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$$

and equivalent to the system of linear equations

$$(3) \quad \begin{cases} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \end{cases}$$

Remark The equation (1) (or (2) or (3)) has a solution if and only if b is a linear combination of the columns of A (or, in other words b is in the span of columns of A)

Theorem 4 The following are equivalent:

(a) Equation (1) has a solution for every b in $\mathbb{R}^m$

(b) Any vector in $\mathbb{R}^m$ is a linear combination of the columns of A

(c) Columns of A span $\mathbb{R}^m$

(d) A has a pivot position in each row