Lecture 6 Math 225

Note Title 1/30/2008

1.5 Solution sets of linear systems

Previously: – You learned how to solve a system of linear equations:

Example 1

$$(1)\quad \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 + 5x_2 + 6x_3 = 0 \end{cases}$$

– You learned that (1) has the same solutions as:

$$(2)\quad x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or as:

$$(3)\quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This lecture will not give a new algorithm for solving (1) but it will organise its solutions

Example 1 (solution)

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

nonpivot column
$\Rightarrow x_3$ is free

$$\begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_3 &= \text{free} \end{aligned} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \quad - \text{free} \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{in } \mathbb{R}.$$

$\Rightarrow$ Solution set $= \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ a line.

In general: The homogeneous equation

$$Ax = 0$$

has the solution set Span $\{v_1, v_2, \ldots, v_n\}$ where the number of vectors, n, is equal to the number of free variables.

Example 2

(4) $\qquad x_1 + 2x_2 + 3x_3 = 0$

$$\Rightarrow \begin{cases} x_1 = -2x_2 - 3x_3 \\ x_2 = \text{free} \\ x_3 = \text{free} \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \qquad x_2, x_3 \text{ in } \mathbb{R}$$

$\Rightarrow$ Solution set is Span $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

a plane

Remark (4) is called the implicit equation of the plane while:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad s, t \text{ in } \mathbb{R}$$

is called the parametric equation of the plane

Nonhomogeneous Equations : $Ax = b \quad b \neq 0$

If the equation has at least one solution then all solutions are of the form

$$x = p + v_h$$

where p is a solution of $Ax = b$ and v_h is any solution of $Ax = 0$.

Example 1.1 $\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 4x_1 + 5x_2 + 6x_3 = 9 \end{cases}$

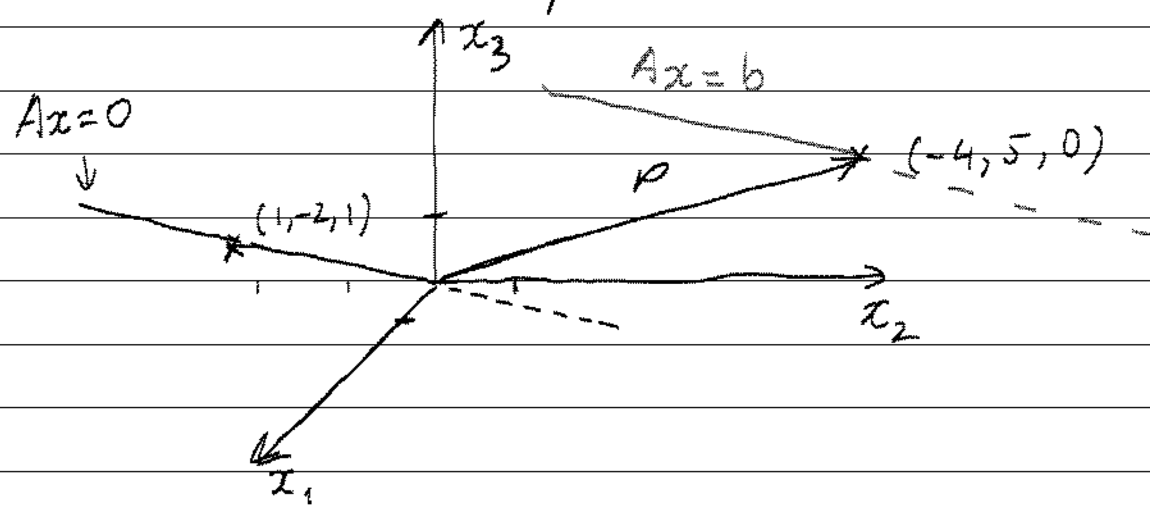
$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -3 & -6 & -15 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

nonpivot column
x_3 is free

$$\begin{aligned} x_1 &= -4 + x_3 \\ x_2 &= 5 - 2x_3 \\ x_3 &= \text{free} \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 \\ 5 \\ 0 \end{bmatrix}}_{p} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Geometric (graphic) representation:



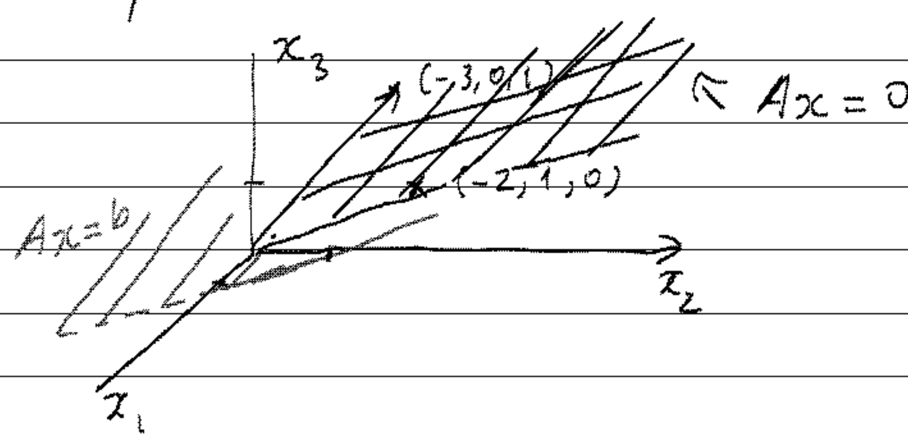
The solution set of $Ax=b$ is the translation by p of the solution set of $Ax=0$. The line $Ax=b$ is parallel to the line $Ax=0$.

Example 2.1 $\quad x_1 + 2x_2 + 3x_3 = 1$

$\Rightarrow x_1 = 1 - 2x_2 - 3x_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underset{\uparrow \atop p}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Geometric representation



The plane $Ax = b$ is parallel to $Ax = 0$ and translated by 1 unit in the x_1 direction (translated by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$).