1.7. Linear dependence and independence

Def: The vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^m$ are linearly dependent if there exists numbers $x_1, x_2, \ldots, x_n$ not all zero such that

$$x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = 0$$

If vectors $a_1, a_2, \ldots, a_n$ are not linearly dependent then they are called linearly independent

Example 1 Check whether

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

are linearly dependent or not.

Solution : We are looking for x_1, x_2 and x_3 such that :

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a system of linear equations with extended matrix

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ non pivot column

$\Rightarrow$ x_3 is free $\Rightarrow$ there are non-zero solutions (for example choose $x_3 = 1$ and solve for x_1 and x_2).

$\Rightarrow$ the three vectors are <u>linearly dependent</u>.

<u>Example 2</u> Are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

linearly independent?

Solution: Look for x_1, x_2, x_3 in $\mathbb{R}$ such that

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 0$$

The extended matrix of the system is:

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$

pivot columns

The system has a unique sln $x_1 = 0, x_2 = 0, x_3 = 0$ hence the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

are linearly independent!

Remarks Given vectors $a_1, a_2, \dots, a_n$ in $\mathbb{R}^m$ form the matrix A with columns $a_1, a_2, \dots, a_n$:

$$A = [a_1, a_2, \dots, a_n].$$

$a_1, a_2, \dots, a_n$ are linearly independent if and only if

$$(*) \qquad Ax = 0$$

has a unique solution ($x = 0$).

Consequently if $m < n$ (the number of components of the vectors $a_1, \dots, a_n$ is less than the number of vectors) then $a_1, a_2, \dots, a_n$ are linearly dependent. This is because the extended matrix of $(*)$:

$$[A, 0]$$

cannot have all first n columns pivot columns. It can have at most m leading entries hence at most m pivot columns.

Theorem: Vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^m$ are linearly dependent if and only if at least one vector is a linear combination of the others. In fact if $a_1 \neq 0$ then some a_j, $j > 1$ is a linear combination of the preceding vectors.

Corollary: If a set of vectors contain the 0 vector then the set is linearly dependent.

This is because the zero vector is a linear combination of the others with zero weights.

The case of two vectors: u, v in $\mathbb{R}^m$ are linearly dependent if and only if one is a multiple of the other.

The case of three vectors: If u, v in $\mathbb{R}^m$ are linearly independent then u, v, w are linearly dependent if and only if

w is in Span$\{u, v\}$