Lecture 8 Math 225

Note Title 2/6/2008

Section 2.1. Operation with matrices

Labeling: The (i, j)- entry in a matrix is the number on its i^{th} row and j^{th} column.

The notation $A = [a_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

means that A is an $m \times n$ matrix and its (i, j) entry has been labeled a_{ij}. If the number of rows and columns is clear from the context then we use

$$A = [a_{ij}].$$

Def Two matrices are equal if they have the same size and their corresponding entries are equal:

$$A = [a_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}} \qquad B = [b_{ij}]_{\substack{1 \le i \le m' \\ 1 \le j \le n'}}$$

$A = B$ if and only if $m = m'$, $n = n'$ and $a_{ij} = b_{ij}$ for all i, j.

<u>Addition of matrices</u> : For two matrices of the same size one adds the corresponding entries :

A, B are both $m \times n$ matrices :

$$A = [a_{ij}], \; B = [b_{ij}] \text{ then}$$

$$A + B = [a_{ij} + b_{ij}]$$

<u>Example</u>

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} =$$

$$= \begin{bmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

<u>General properties of addition of matrices</u>:
A, B, C are all $m \times n$ matrices:

$A + B = B + A$

$(A + B) + C = A + (B + C)$

$A + O = O + A = A$ where O is the $m \times n$ matrix with all entries zero.

Multiplying matrices by scalars: one multiplies each entry of the matrix by the scalar

$A = [a_{ij}]$, r in $\mathbb{R}$ then

$$rA = [r\,a_{ij}]$$

Example 2 $(-1)\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix}$

Notation $(-1)A = -A$.

General properties of multiplication by scalars:

r, s are scalars (numbers) A, B are both $m \times n$ matrices

$$r(sA) = (rs)A$$
$$(r+s)A = rA + sA$$
$$r(A+B) = rA + rB$$

Product of two matrices

Recall the product of a matrix with a vector

$$A = [a_1, a_2, \dots, a_n] \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

If $a_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ $j = 1, 2, \dots, n$ then

$A = [a_{ij}]$ and the i^{th} component of Ax

$$[Ax]_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$$

__Def__ If $A = [a_1, a_2, \dots, a_n]$ is $m \times n$ and $B = [b_1, b_2, \dots, b_p]$ is $n \times p$ then

$AB = [Ab_1, Ab_2, \dots, Ab_p]$ is an $m \times p$ matrix called the product of A and B.

<u>Remark</u> The definition above is the only one for which the following works:

$$A(Bx) = (AB)x \quad \text{for any } x \text{ in } \mathbb{R}^p$$

<u>Example</u>
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} =$$

$$= \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right]$$

$$= \left[(-1)\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 6 \end{bmatrix}, 2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right]$$

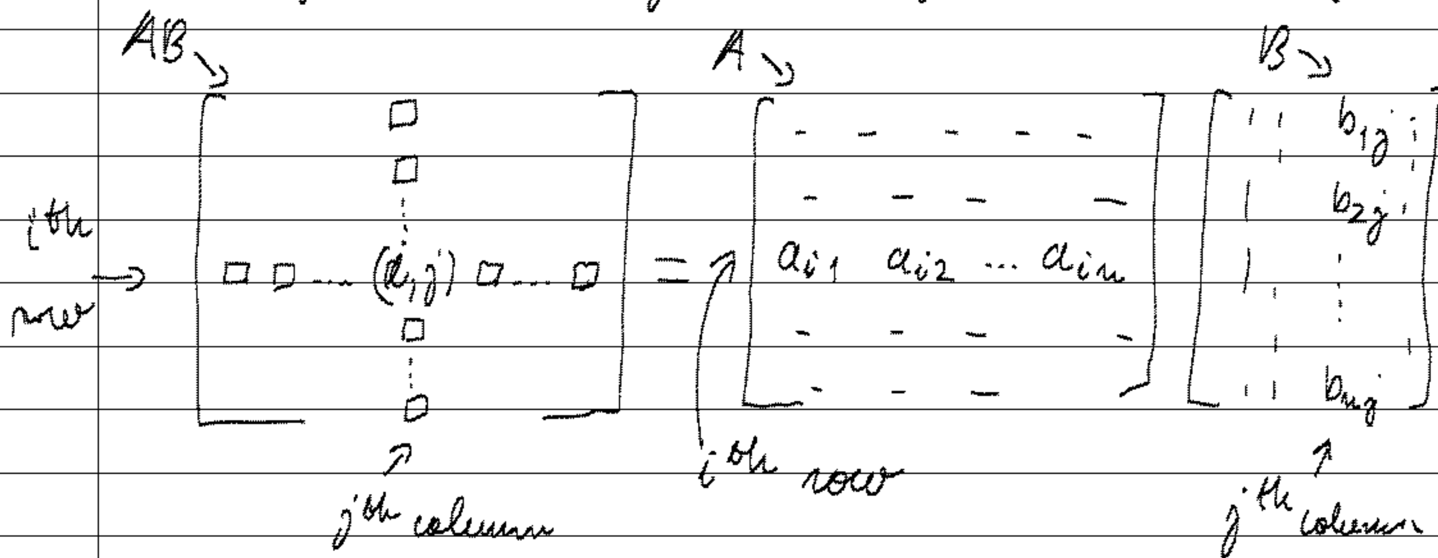
$$= \begin{bmatrix} 2 & 20 \\ 2 & 47 \end{bmatrix}$$

<u>Another way the row-column rule:</u>

$$A = [a_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}} \qquad B = [b_{kl}]_{\substack{1 \le k \le n \\ 1 \le l \le p}}$$

then the (i,j)-entry of AB is

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$



<u>Example</u> $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1(-1)+2\cdot 0+3\cdot 1 & 1\cdot 2+2\cdot 3+3\cdot 4 \\ 4\cdot(-1)+5\cdot 0+6\cdot 1 & 4\cdot 2+5\cdot 3+6\cdot 4 \end{bmatrix} = \begin{bmatrix} 2 & 20 \\ 2 & 47 \end{bmatrix}$$

General properties of product of matrices

$A(BC) = (AB)C$ A is $m \times n$, B is $n \times p$, C is $p \times q$

$A(B+C) = AB + AC$ A is $m \times n$, B & C are $n \times p$

$(B+C)A = BA + CA$ B & C are $m \times n$, A is $n \times p$

$r(AB) = (rA)B = A(rB)$ r in $\mathbb{R}$, A is $m \times n$
B is $n \times p$

$I_m A = A = A I_n$ A is $m \times n$.

I_m (and I_n) are $m \times m$ (respectively $n \times n$) with all entries zero except (i,i), $1 \le i \le m$ $(1 \le i \le n)$.

Facts 1° For square matrices A, B (number of rows = number of columns) both AB and BA can be computed. In general $AB \neq BA$. When $AB = BA$ we say that A & B commute.

2° $AB = AC \not\Rightarrow B = C$

3° $AB = 0 \not\Rightarrow A$ or B are zero.