Lecture 9 Math 225

Note Title 2/11/2008

Summary

Sect 2.1 (Cont.) Powers and Transpose of a matrix

Sect. 2.2 The inverse of a matrix

2.1. (Continuation)

Powers of a square matrix: If A is an $n \times n$ matrix (square matrix) then by convention

$$A^0 = I_n \,; \quad A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$$ where k is a natural number

The transpose of a matrix

Def If A is $m \times n$ matrix then its transpose, denoted by A^T, is a $n \times m$ matrix whose columns are formed by the corresponding rows of A (the i^{th} column of A^T is the i^{th} row of A, for any $i = 1, 2, \dots, m$)

Example $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Equivalently : the (i,j) entry of A^T is the (j,i) entry of A.

Example $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

The $(3,2)$ entry of A^T $= (2,3)$ entry of A
$= 7$!

Properties

a) $(A^T)^T = A$
b) $(A+B)^T = A^T + B^T$
c) $(rA)^T = rA^T$ for any scalar r
d) $(AB)^T = B^T A^T$

2.2. The inverse of a matrix

Recall that $AB = AC$ does not imply in general that $B = C$. In particular

$$AB = O$$

does not imply in general $B = O$. These implications become true for A invertible.

<u>Def</u> An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix C such that

$$CA = I_n \quad \text{and} \quad AC = I_n$$

C satisfying above is called an inverse of A.

<u>Remark</u> The inverse, if it exists, is unique.

<u>Notation</u> The unique inverse of A, if it exists, is denoted by A^{-1}.

<u>Example</u> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $ad - bc \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$ then A is not invertible

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}^{-1}$ does not exist!

<u>Remark</u> If $AB = AC$ and A is invertible then $B = C$:

$$AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$$
$$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$$
$$\Rightarrow I_n B = I_n C \Rightarrow B = C$$

<u>Theorem</u> If A is $n \times n$ and invertible then for any b in $\mathbb{R}^n$ the matrix eq:

$$(*) \qquad Ax = b$$

has a unique solution

$$x = A^{-1}b.$$

Proof $A^{-1}b$ is a solution because substituting $A^{-1}b$ for x in left hand side of (*) we get an identity:

$$A(A^{-1}b) = (AA^{-1})b = I_n b = b$$

For any other solution x we have

$$\begin{aligned} Ax = b &\Rightarrow A^{-1}(Ax) = A^{-1}b \\ &\Rightarrow (A^{-1}A)x = A^{-1}b \\ &\Rightarrow I_n x = A^{-1}b \\ &\Rightarrow x = A^{-1}b. \end{aligned}$$

So $A^{-1}b$ is the only solution.

Example: Solve $\begin{cases} x_1 + 2x_2 = 5 \\ 3x_1 + 4x_2 = 6 \end{cases}$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$$