Math 225

Problems for Review 2

0. Study your notes and the textbook (Sects. 3.1–3.3, ~~4.1–4.3, 4.5–4.6~~). and Sect 2.1–2.3

1. (a) Give the definition of the determinant $\det A$ of an $n \times n$ matrix A, where $n \geq 1$.

   (b) Show that if $A = (a_{ij})_{n\times n}$ is an $n \times n$ matrix with integer entries a_{ij} then $\det A$ is also an integer.

2. (a) Explain what happens to $\det A$ when an elementary row (column) operation is applied to A.

   (b) Evaluate $\det E$ if E is an elementary matrix.

   (c) Suppose that B is a matrix which is row equivalent to A and $\det A = 2$. Find (if possible) $\det B$.

3. Evaluate $\det A$ if A is the following matrix

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -3 & 2 & -5 & 13 \\ 1 & -2 & 10 & 4 \\ -2 & 9 & -8 & 25 \end{bmatrix};\quad \text{(b)}\quad A = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ .. & .. & \dots & .. & .. \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{n\times n},\ n = 2, 3, 4, \dots$$

4. Evaluate $\det A$ if (a) $A = \begin{bmatrix} 1 & n & \dots & n & n \\ n & 2 & \dots & n & n \\ .. & .. & \dots & .. & .. \\ n & n & \dots & n-1 & n \\ n & n & \dots & n & n \end{bmatrix}_{n\times n}$; (b) $A = \begin{bmatrix} b & a & \dots & a & a \\ a & b & \dots & a & a \\ .. & .. & \dots & .. & .. \\ a & a & \dots & b & a \\ a & a & \dots & a & b \end{bmatrix}_{n\times n}$.

5. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ be columns of an $n \times n$ matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \dots\ \mathbf{a}_n]$ and $\det A = 2$. Evaluate the following determinants

   (a) $\det[\mathbf{a}_n\ \mathbf{a}_1\ \mathbf{a}_3\ \dots\ \mathbf{a}_{n-1}]$;

   (b) $\det[\mathbf{a}_1\ 2\mathbf{a}_2\ \dots\ (n-1)\mathbf{a}_{n-1}\ n\mathbf{a}_n]$;

   (c) $\det[\mathbf{a}_1\ \mathbf{a}_1 + \mathbf{a}_2\ \mathbf{a}_2 + \mathbf{a}_3\ \dots\ \mathbf{a}_{n-1} + \mathbf{a}_n]$;

   (d) $\det[\mathbf{a}_1 - \mathbf{a}_2\ \mathbf{a}_2 - \mathbf{a}_3\ \dots\ \mathbf{a}_n - \mathbf{a}_1]$.

6. Find (if possible) $\det 2A$, $\det(-B)$, $\det(A+B)$, $\det(ABA^2B^2)$, $\det(\operatorname{adj} A)$, $\det(\operatorname{adj} B)$ if A, B are matrices of size $n \times n$ with $\det A = 1$ and $\det B = 2$.

7. Show that

   (a) An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

   (b) If A, B are $n \times n$ matrices, then $\det AB = \det A \det B$.

8. (a) Write down formulas for the adjoint $\operatorname{adj} A$, and the inverse A^{-1} of a matrix A.

   (b) ~~Explain and prove the false expansion formula and use this formula to~~ show that $\operatorname{adj} A \cdot A = A \cdot \operatorname{adj} A = \det A \cdot I_n$.

9. Find (if possible) the adjoint $\operatorname{adj} A$ if (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$; (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

10. Explain Cramer's rule and, using this rule (if possible), solve SLE $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = [1, 0, 1]^T$.