Math 225 Section M1 Final Exam

May 8, 2009

1. Consider the system of linear equations:

$$\begin{aligned} x_2 + 2x_3 &= 3 \\ 3x_1 + 4x_2 + 5x_3 &= 12 \\ 3x_1 + 5x_2 + 7x_3 &= 15 \end{aligned} \tag{1}$$

   (a) (5 points) Write the above system as a matrix equation and as a vector equation.

   (b) (5 points) Find all solutions of the system (1) and write them in (parametric) vector form.

   (c) (5 points) Let A be the matrix of the system (1). What is the dimension of Nul A? Justify your answer.

   (d) (5 points) Are the columns of A linearly independent? Justify your answer.

   (e) (5 points) What is the rank of A? Justify your answer.

2. Consider the 4×4 matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 6 & 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

   (i) (10 points) Is any of A and B invertible? Justify your answer. Pick one that is invertible and calculate the $(2,4)$ entry of its inverse.

   (ii) (5 points) Is AB invertible? Justify your answer.

   (iii) (5 points) Do the rows of AB span $\mathbb{R}^4$? Justify your answer.

3. Let A be a square $n \times n$ matrix.

   (a) (5 points) Assume $\det(A^T A) = 0$. Show that A is not invertible.

   (b) (5 points) Assume that $A^T A x = 0$ has exactly one solution. Show that A is invertible.

4. Consider the set S of all polynomials, $p(t)$, of degree at most three with the property

$$p(-1) = 0.$$

   (i) (10 points) Is S a subspace of $\mathbb{P}_3$? Justify your answer.

   (ii) (10 points) Let $p_1(t) = 1 + t$, $p_2(t) = -1 + t^2$ and $p_3(t) = 1 + t^3$. Is $\{p_1, p_2, p_3\}$ a basis for S? What is the dimension of S? Justify your answers.

5. Consider the 3×3 matrix:

$$A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ -4 & 7 & -2 \end{bmatrix}$$

   (a) (5 points) Find all eigenvalues of A. Note that 1 should be an eigenvalue.

   (b) (5 points) Find a basis for each eigenspace of A.

   (c) (5 points) Is A diagonalizable? Justify your answer. If it is diagonalizable find a diagonal matrix D and an invertible matrix P such that:

$$D = P^{-1} A P.$$

6. (10 points) Let

$$B = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

   Calculate B^{2009}.

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4. (a)
$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 15 \end{bmatrix}$$ — matrix form

$$x_1 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 15 \end{bmatrix}$$ — vector form

(b) Use row reduction of the extended matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 12 \\ 3 & 5 & 7 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 & 12 \\ 0 & 1 & 2 & 3 \\ 3 & 5 & 7 & 15 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 3 & 4 & 5 & 12 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 & 12 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 3 & 0 & -3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is free.

Sln is

$$\begin{cases} x_1 = x_3 \\ x_2 = 3 - 2x_3 \\ x_3 = x_3 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{\text{(parametric) vector form.}} \quad x_3 \text{ arbitrary}$$

(c) $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$, from the vector form above

the solution of $Ax = 0$ is span$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. so

$\dim(\text{Nul } A) = 1$

(d). Since Nul $A \neq \{0\}$ the columns of A <u>cannot</u> be linearly independent.

(e) $\dim(\mathrm{Nul}\,A) + \mathrm{rank}\,A = 3$ by the rank theorem

From (c) we have $\dim(\mathrm{Nul}\,A) = 1 \Rightarrow \mathrm{rank}\,A = 2$.

2. (i) $\det A = 1 \cdot 3 \cdot 0 \cdot 9 = 0 \Rightarrow A$ is <u>not invertible</u>

$\det B = 1 \cdot 5 \cdot 8 \cdot 10 = 400 \neq 0 \Rightarrow B$ <u>is invertible</u>.

$$\left(B^{-1}\right)_{(2,4)} = \frac{1}{\det B} B_{42} = \frac{1}{\det B}(-1)^{4+2}\begin{vmatrix} 1 & 3 & 4 \\ 0 & 6 & 7 \\ 0 & 8 & 9 \end{vmatrix}$$

$$= \frac{1}{400}(6 \cdot 9 - 8 \cdot 7) = \frac{1}{400}(54 - 56) = -\frac{2}{400}$$

$$= -\frac{1}{200}$$

(ii) $\det(AB) = (\det A)(\det B) = 0 \cdot 400 = 0$

$\Rightarrow AB$ is <u>NOT invertible</u>

(iii) Since AB is not invertible, by the inverse matrix theorem, the rows of AB <u>cannot span</u> $\mathbb{R}^4$.

3 (a) $\det(A^TA) = (\det A^T)(\det A) = (\det A)^2$

So $(\det A)^2 = 0 \Rightarrow \det A = 0 \Rightarrow A$ is <u>NOT INVERTIBLE</u>

(b) $(A^TA)x = \vec{0}$ ~~implies~~ has exactly one sln implies A^TA is invertible (by the inverse matrix Th). $\Rightarrow \det(A^TA) \neq 0 \Rightarrow (\det A)^2 \neq 0$

$\Rightarrow (\det A) \neq 0 \Rightarrow A$ is invertible

4 (i) Let p_1, p_2 be in S then

$$(p_1 + p_2)(-1) = p_1(-1) + p_2(-1) = 0 + 0 = 0$$

So $p_1 + p_2$ is in S

Let p in S and λ in $\mathbb{R}$ then

$(\lambda p)(-1) = \lambda p(-1) = \lambda \cdot 0 = 0$ so (λp) is in S

Since S is closed under addition and multiplication (by scalars), S is a subspace of P_3

4 (ii) Linear independence:

$$c_1 p_1(t) + c_2 p_2(t) + c_3 p_3(t) = \underbrace{0 + 0\cdot t + 0t^2 + 0\cdot t^3}_{\text{the zero polynomial}}$$

$$\Leftrightarrow (c_1 - c_2 + c_3) + c_1 t + c_2 t^2 + c_3 t^3 = 0 + 0t + 0t^2 + 0t^3$$

$\Leftrightarrow$ $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$. So the only linear comb of p_1, p_2, p_3 that gives the zero polynomial is the one with all weights zero. $\Rightarrow$ p_1, p_2, p_3 are linearly independent

Spanning property:

If p is in S then $\begin{cases} p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ p(-1) = a_0 - a_1 + a_2 - a_3 = 0 \end{cases}$

Consider $a_1 p_1(t) + a_2 p_2(t) + a_3 p_3(t) = \cancel{(a_1 - a_2 + a_3) + a_1 t} + a_2 t$

$$= \underbrace{(a_1 - a_2 + a_3)}_{= a_0} + a_1 t + a_2 t^2 + a_3 t^3$$

$= p(t)$. So p_1, p_2, p_3 span S

Because p_1, p_2, p_3 span S and are linearly indep

they do form a basis for S

dim S = # of vectors in a basis = 3.

5 (a) $0 = \det(A - \lambda I_3) = \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ -4 & 7 & -2-\lambda \end{vmatrix}$

Cofactor expansion w.r.t 2nd row $= (1-\lambda)\begin{vmatrix} 5-\lambda & 3 \\ -4 & -2-\lambda \end{vmatrix}$

$$= (1-\lambda)\left[\lambda^2 - 3\lambda - 10 + 12\right] = (1-\lambda)\left[\lambda^2 - 3\lambda + 2\right]$$

$$= (1-\lambda)(\lambda-1)(\lambda-2)$$

Eigenvalues are ~~given by 1-λ~~ $\lambda_1 = \lambda_2 = 1 \quad \lambda_3 = 2$

(b) $\text{Nul}\left[A - 1I_3\right] = \text{Nul}\begin{bmatrix} 4 & -2 & 3 \\ 0 & 0 & 0 \\ -4 & 7 & -3 \end{bmatrix}$

$$\begin{bmatrix} 4 & -2 & 3 \\ 0 & 0 & 0 \\ -4 & 7 & -3 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 3 \\ -4 & 7 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3/4 \\ 0 \\ 1 \end{bmatrix}$$

x_3 is free

$$\Rightarrow \operatorname{Nul}\left[A - 1I_3\right] = \operatorname{Span}\left\{ \begin{bmatrix} -3/4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\operatorname{Nul}\left[A - 2I_3\right] = \operatorname{Nul}\begin{bmatrix} 3 & -2 & 3 \\ 0 & -1 & 0 \\ -4 & 7 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 3 \\ 0 & -1 & 0 \\ -4 & 7 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 3 \\ 0 & -1 & 0 \\ 0 & 13/3 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

x_3 is free

$$\Rightarrow \operatorname{Nul}\left[A - 2I_3\right] = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(C) There are only two linearly independent eigenvectors (in the bases for all eigenspaces) so A is <u>NOT diagonalizable</u>

6. First diagonalize B:

* e-values: $\begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} = 0$

$$\Leftrightarrow \quad \lambda^2 - 4 + 3 = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

* e-vectors.

$\lambda_1 = 1$: $\text{Nul} \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 = 3x_2 \\ x_2 = x_2 \end{matrix}$$

x_2 is free

$\lambda_2 = -1$: $\text{Nul} \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix}$$

x_2 is free

So $\cancel{B}$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = P^{-1} B P$

with $P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ and $P^{-1} = \frac{1}{\det P} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

Then $B^{2009} = P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{2009} P^{-1}$

$$= P \begin{bmatrix} 1^{2009} & 0 \\ 0 & (-1)^{2009} \end{bmatrix} P^{-1}$$

$$= P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

So $B^{2009} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$