Math 225 Section M1 Midterm II

April 2, 2009

Solutions

1. Let A be a $n \times n$ matrix. Determine whether the next statements are True or False. No explanation needed.

   (a) (5 points) If $Ax = 0$ has more than one solution then A is invertible.

   (b) (5 points) If A is not invertible then A does not have a pivot position in each row.

   (c) (5 points) If the rows of A do not span $\mathbb{R}^n$ then the matrix is invertible.

   (d) (5 points) If there exists a vector b in $\mathbb{R}^n$ such that $Ax = b$ has no solution then A is not invertible.

2. (10 points) Consider the matrix A and vector b below:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 24 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   Calculate, if possible, $A^{-1}b$.

3. (i) (5 points) What is the definition of

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}?$$

   (ii) (5 points) Let A be a 4×4 matrix $A = [a_1,\ a_2,\ a_3,\ a_4]^T$ (the vectors $a_1,\ a_2,\ a_3,\ a_4$ are the rows of A) with $\det A = -3$. Calculate $\det[a_2,\ a_1,\ 3a_3 - 2a_4,\ a_4]^T$.

   (iii) (5 points) Let A be as in part (ii) and B be another 4×4 matrix with $\det B = 2$. Calculate $\det(A^{-1}B)$.

   (iv) (5 points) Calculate the determinant of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & -6 & 4 \\ -3 & 1 & -4 & -4 \\ 5 & 0 & 0 & 10 \\ -1 & -3 & 9 & -2 \end{bmatrix}.$$

   (v) (5 points) If the matrix A in part (iv) above is invertible find its $(4, 2)$ entry.

4. Consider the system of linear equations:

$$\begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b.$$

   (a) (10 points) Does Cramer rule apply to this system of equations. Justify your answer.

   (b) (5 points) Find x_1 when $b = [0,\ 1,\ 2]^T$.

(c) (5 points) Find x_2 when $b = [1,\ 2,\ 0]^T$.

(d) (5 points) Find x_1 when $b = [1,\ 0,\ 1]^T$.

5. (10 points) Suppose $AB = AC$ where $B,\ C$ are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?

6. (10 points) Let A be an $m \times n$ matrix and B an $n \times m$ matrix such that $AB = \mathbb{I}_m$. Show that the columns of A span $\mathbb{R}^m$ and the columns of B are linearly independent.

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Solution page

1 (a) F

(b) T

(c) F

(d) T

2. $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 4 & 9 & 2 \\ 1 & 8 & 24 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 6 & 21 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 3 & 1/2 \\ 0 & 0 & 1 & -1/3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -1/3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -1/3 \end{array}\right]$$

$$A^{-1} b = \begin{bmatrix} -1 \\ 3/2 \\ -1/3 \end{bmatrix} \qquad A^{-1} = \begin{pmatrix} 4 & -4 & 1 \\ -5/2 & 7/2 & -1 \\ 2/3 & -1 & 1/3 \end{pmatrix}$$

$\det A = 6.$

3. (i)

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(ii) $\det\left[a_2, a_1, 3a_3 - 2a_4, a_4\right]^T = -3 \det A = 9$

because rows 1 and 2 were swapped, row 3 was multiplied by 3 and then replaced by itself minus 2 row 4.

(iii) $\det(A^{-1}B) = \det(A^{-1}) \det B = \dfrac{1}{\det A} . \det B = -\dfrac{2}{3}$

(iv)

$$\begin{vmatrix} 2 & 3 & -6 & 4 \\ -3 & 1 & -4 & -4 \\ 5 & 0 & 0 & 10 \\ -1 & -3 & 9 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -6 & 0 \\ -3 & 1 & -4 & +2 \\ 5 & 0 & 0 & 0 \\ -1 & -3 & 9 & 0 \end{vmatrix}$$

$$= 5\begin{vmatrix} 3 & -6 & 0 \\ 1 & -4 & 2 \\ -3 & 9 & 0 \end{vmatrix} = 5 \cdot 2(-1)\begin{vmatrix} 3 & -6 \\ -3 & 9 \end{vmatrix}$$

$$= -10 \cdot 9 = -90$$

(vi) $\det A \neq 0 \Rightarrow A$ is invertible

$$(A^{-1})_{(4,2)} = \frac{1}{\det A} \cdot C_{24} = \frac{1}{\det A} \begin{vmatrix} 2 & 3 & -6 \\ 5 & 0 & 0 \\ -1 & -3 & 9 \end{vmatrix}$$

$$= \frac{1}{-90} \cdot (-5) \begin{vmatrix} 3 & -6 \\ -3 & 9 \end{vmatrix} = \frac{1}{-90} (-5) \cdot 9$$

$$= \frac{1}{2}$$

4 a. $$\begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ -2 & 0 & 2 \end{vmatrix} = (-1) \begin{vmatrix} -3 & 1 \\ -2 & 2 \end{vmatrix}$$

$$= (-1)(-4) = 4$$

$\det \neq 0 \Rightarrow$ Cramer Rule applies

b. $$x_1 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{vmatrix} \Big/ 4 = 0$$ (first & last columns are equal.

c. $$x_2 = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix} \Big/ 4 = 2 \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} \Big/ 4 = 7/2$$

4 d. $x_1 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} \Big/ 4 = 0$ first & second columns are equal

5. $AB = AC$

$\Rightarrow \quad A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow \quad (A^{-1}A)B = (A^{-1}A)C$

$\Rightarrow \quad I_n B = I_n C$

$\Rightarrow \quad B = C$

No, if A is not invertible, say $A = O_n$ (all entries are zero) and $B \neq C$ are arbitrary then

$AB = AC = O_{n \times p}$ but $B \neq C$.

6. $AB = \mathbb{I}_m$ then

$B x = \vec{0}$ can only have $x = \vec{0}$ as a solution

otherwise $(AB)x = I_m x$

$\Rightarrow A(Bx) = x$

$\Rightarrow A \cdot \vec{0} = x$

$\Rightarrow \vec{0} = x$ contradiction

So $Bx = \vec{0}$ has only $\vec{0}$ as a sln $\Rightarrow$ columns of B are linearly independent

Let $\vec{b} \in \mathbb{R}^m$ be such that $Ay = \vec{b}$ is incompatible (no sln's exist). Then ~~for any x in~~

$(AB)\vec{b} = \mathbb{I}_m \vec{b}$

$\Rightarrow A(B\vec{b}) = \vec{b} \Rightarrow y = B\vec{b}$ should have been a sln of $Ay = \vec{b}$ Contradiction

So $Ay = \vec{b}$ is always solvable for $\vec{b}$ in $\mathbb{R}^m$

$\Rightarrow$ columns of A span $\mathbb{R}^m$