Math 225 Section M1 Midterm I

February 19, 2009

1. Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 4 \\ 3 & 4 & 10 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}.$$

   (i) (10 points) Find all solutions of $Ax = b$. vector form

   (ii) (10 points) Write the solutions of $Ax = b$ in parametric form.

   (iii) (10 points) What is the geometric interpretation of the solutions of $Ax = 0$ and $Ax = b$?

2. Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \ v_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \ v_3 = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}, \ v_4 = \begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}, \ b_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

   (a) (10 points) Is b_1 in Span$\{v_1, v_2, v_3, v_4\}$? Justify your answer.

   (b) (10 points) Is b_2 in Span$\{v_1, v_2, v_3, v_4\}$? Justify your answer.

   (c) (5 points) Do v_1, v_2, v_3, v_4 span $\mathbb{R}^3$? Justify your answer

   (d) (5 points) Do v_1, v_2, v_3, v_4, b_2 span $\mathbb{R}^3$? Justify your answer

3. Let

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \ a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \ a_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \ a_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ a_5 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

   (i) (5 points) Are a_1, a_2, a_3, a_4 linearly dependent? Justify your answer.

   (ii) (5 points) Are a_1, a_2, a_3, a_4, a_5 linearly independent ? Justify your answer.

4. Let

$$A = \begin{bmatrix} -2 & -1 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \ B = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}.$$

   (a) (5 points) Calculate $2B - 2A$.

   (b) (5 points) Find the $(3, 2)$ entry of AB.

5. (10 points) Write the reduced echelon form of a 3×3 matrix A such that the first two columns of A are pivot columns and

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

6. In (i) and (ii) suppose that the vectors are linearly independent. What can you say about the numbers $a, \ldots, f$? Justify your answers.

(i) (5 points) $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ (ii) (5 points) $\begin{bmatrix} a \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \\ 1 \end{bmatrix}$.

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Solution page

1. Extended (augmented) matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 3 & 4 & 10 & 7 \end{bmatrix} \underset{\text{and row 2}}{\overset{\text{swap row 1}}{\sim}} \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 1 & 1 \\ 3 & 4 & 10 & 7 \end{bmatrix}$$

$$\overset{\text{row 3 = row 3 - 3 row 1}}{\sim} \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 \end{bmatrix}$$

$$\overset{\text{row 3 = row 3 + 2 row 2}}{\sim} \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns non-pivot columns

$x_1 = 3 - 2x_2 - 4x_3 = 1 - 2x_3$

$\Rightarrow x_2 = 1 - x_3$

$x_3 =$ free

$\Rightarrow$ System is consistent and x_3 is free

Answer for (i) $x_1 = 1 - 2x_3$
$x_2 = 1 - x_3$
x_3 is free.

1.(ii) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \quad x_3 \text{ in } \mathbb{R}$$

1.(iii) Sln's of $A\vec{x} = 0$ are given by:
$$\vec{x} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \quad x_3 \text{ in } \mathbb{R}$$
and form a line (along the vector $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$) which passes through the origin.

Sln's of $A\vec{x} = \vec{b}$ are given by formula in 1.(ii) and form a line parallel to the one described above ~~and shifted by~~ and passing through $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

2. (a) We are looking for weights x_1, x_2, x_3, x_4 such that

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Extended matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 & 1 \\ 2 & 3 & 6 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

System is consistent

nonpivot

$\Rightarrow$ b_1 <u>is</u> in $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$!

2. (b) Extended matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 & 1 \\ 2 & 3 & 6 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot column

System is inconsistent $\Rightarrow$ b_2 is NOT in $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$.

~~2(C) Since b_2 is missing from Span$\{v_1, v_2, v_3, v_4\}$~~

2(C) $\{v_1, v_2, v_3, v_4\}$ span $\mathbb{R}^3$ if any vector with three entries is a linear combination of v_1, v_2, v_3, v_4. ~~b_2~~ b_2 is NOT so $\{v_1, v_2, v_3, v_4\}$ DO NOT span $\mathbb{R}^3$

2(d) $\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 & 1 \\ 2 & 3 & 6 & 7 & 0 \end{bmatrix} \underset{\text{See 2(b)}}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

leading entry in each ROW $\Rightarrow$ v_1, v_2, v_3, v_4, b_2 span $\mathbb{R}^3$.

3 (i) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ all columns are pivot columns

$\Downarrow$

a_1, a_2, a_3, a_4 are linearly independent

So No! they are not linearly dependent!

(ii) There are more vectors (5) than entries (4).

So a_1, a_2, a_3, a_4, a_5 are linearly dependent.

Answer NO! they are not linearly independent

4. (a) $2B - 2A = 2(B-A) = 2\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

(b) $(3,2)$ entry is obtained by multiplying the entries in the 3^{rd} row of A with corresponding entries in the 2^{nd} column of B

$(3,2)$ entry of $AB = 0\cdot 0 + 1\cdot 1 + 2\cdot 2 + 3\cdot 3 = 1+4+9 = 14$

5. Let A_r be the reduced echelon form of A. Since the sln's of $Ax=0$ do not change by row equivalent operations:

$$(*) \qquad A_r \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since A_r is in reduced echelon form and has first two columns pivot columns $\Rightarrow A_r = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & \gamma \end{bmatrix}$

where α, β, γ, will be determined by plugging in $(*)$:

$$\begin{cases} 1 + 3\alpha = 0 \\ 2 + 3\beta = 0 \\ 3\gamma = 0 \end{cases} \Rightarrow \begin{array}{l} \alpha = -1/3 \\ \beta = -2/3 \\ \gamma = 0 \end{array} \Rightarrow A_R = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

6. (i) $\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$

$a \neq 0$ otherwise first column is non-pivot

$c \neq 0$ otherwise second column is non-pivot

$f \neq 0$ otherwise third column is non-pivot

b, d, e can be anything

(ii) $$\begin{bmatrix} a & b & d \\ 1 & c & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & c & e \\ a & b & d \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & c & e \\ 0 & b-ac & d-ae \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & c & e \\ 0 & 1 & f \\ 0 & b-ac & d-ae \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & c & e \\ 0 & 1 & f \\ 0 & 0 & d-ae-f(b-ac) \\ 0 & 0 & 1 \end{bmatrix}$$

all columns are pivot columns. The vectors are linearly independent no matter what values $a, \dots, f$ have!