Math 225

Problems for Review 1

~~0. Study your notes and the textbook (Sects. 1.1–1.5, 1.7, 2.1–2.3, 2.6).~~

1. Bring the augmented matrix $[A\,|\,\mathbf{b}]$ to a reduced row echelon form and solve two systems of linear equations (SLEs) $A\mathbf{x} = \mathbf{b}$, $A\mathbf{x} = \mathbf{0}$, indicating basic and free unknowns, where

(a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 2 \end{bmatrix}$ and $\mathbf{b} = [1, 1, 3]^T$; (b) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = [1, 1, 0]^T$.

2. Write down solutions of SLE's of Problem 1 in parametric vector form.

3. Explain the relation between solution sets of SLE's $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ and give an example of your choice.

4. (a) Let $\mathbf{x}_1 = [1, 0, 1]^T$ and $\mathbf{x}_2 = [1, 2, 3]^T$ be solutions of a SLE $A\mathbf{x} = \mathbf{b}$. Find (if possible) one more solution.
(b) Let $\mathbf{x}_1 = [1, 1, 1, 1]^T$ and $\mathbf{x}_2 = [3, 1, 2, 5]^T$ be solutions of a SLE $A\mathbf{x} = \mathbf{b}$. Find (if possible) one more solution.

5. Compute (if possible) the following matrices (a) $A(B + C) + 2C(B^T - A)$;
(b) $(A^{2007} - B^2)C^T$; (c) $AC - CA$, where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. (a) Give the definition of linearly independent (and dependent) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$.
(b) Explain how to find out whether given vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ are linearly independent and give an example of your choice.

7. Determine whether following vectors are linearly independent
(a) $\mathbf{a}_1 = [1, 2, 3, 0, 1]^T$, $\mathbf{a}_2 = [1, 2, 3, 1, 1]^T$, $\mathbf{a}_3 = [3, 2, 3, 1, 1]^T$, $\mathbf{a}_4 = [2, 4, 6, 1, 3]^T$;
(b) $\mathbf{a}_1 = [1, 2, 3, 0, 1]^T$, $\mathbf{a}_2 = [1, 2, 3, 1, 1]^T$, $\mathbf{a}_3 = [3, 2, 3, 1, 1]^T$;
(c) Columns of the matrices of Problem 1a and 1b.

8. Let $B = [\mathbf{b}_1 \ldots \mathbf{b}_p]$, where $\mathbf{b}_1, \ldots, \mathbf{b}_p$ are columns of matrix B. Show that $AB = [A\mathbf{b}_1 \ldots A\mathbf{b}_p]$.

9. Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix.
(a) Show that if columns of B are linearly dependent then columns of the product AB are also linearly dependent. (Hint: Use Problem 8.)
(b) Show that if columns of the product AB are linearly independent then columns of B are also linearly independent.

10. (a) Give the definition of the span $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$.
(b) Determine whether vector $\mathbf{a}_4 = [2, 4, 6, 1, 3]^T$ lies in the span of $\mathbf{a}_1 = [1, 2, 3, 0, 1]^T$, $\mathbf{a}_2 = [1, 2, 3, 1, 1]^T$, $\mathbf{a}_3 = [3, 2, 3, 1, 1]^T$.

11. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$. Explain how to determine if $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \mathbb{R}^m$ and give an example of your choice.

12. Give the definition of the identity matrix I_n of size $n \times n$ and prove equalities $I_n A = A$, $BI_n = B$, where A has size $n \times m$ and B has size $m \times n$.

13. (a) Give the definition of an inverse of a matrix.
(b) Show that an inverse of a matrix is unique.

14. Let A, B be invertible matrices.
(a) Show that A^{-1}, A^T are invertible and $(A^{-1})^{-1} = A$, $(A^T)^{-1} = (A^{-1})^T$.
(b) Show that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

15. Suppose that E is an elementary matrix of one of three types E_{ij}, $E_i(c)$, $E_{ij}(c)$. Are E^T, E^{-1} elementary matrices? What do E^T, E^{-1} look like?

16. Determine if matrix A is row equivalent to B if

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 4 & 4 & 3 \end{bmatrix}$.

17. Find the inverse A^{-1} (if any) and elementary matrices $E_1, \ldots, E_k$ such that $E_k \ldots E_1 A = I_n$ if

(a) $A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$; (b) $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; (c) $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 3 \end{bmatrix}$.

18. Formulate four equivalent conditions that are all equivalent to the existence of the inverse A^{-1} of an $n \times n$ matrix A. Sketch the proof of equivalence of these conditions.

~~19. Explain the Leontief input-output model and its main production (matrix) equation $\mathbf{x} = C\mathbf{x} + \mathbf{d}$ (or $(I_n - C)\mathbf{x} = \mathbf{d}$).~~

~~20. (a) Determine the production vector $\mathbf{x}$ to satisfy the final demand $\mathbf{d} = [20, 30]^T$ if the consumption matrix is~~ ~~$C =$~~ $\begin{bmatrix} 0.2 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}$.

~~(b) Determine the production vector $\mathbf{x}$ to satisfy the final demand $\mathbf{d} = [40, 60]^T$ if the consumption matrix is~~ ~~$C =$~~ $\begin{bmatrix} 0.2 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}$.

~~(c) Determine the production vector $\mathbf{x}$ to satisfy the final demand $\mathbf{d} = [100, 100, 100]^T$ if the consumption matrix is~~ ~~$C =$~~ $\begin{bmatrix} 0.4 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$.

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Problems for Review 2

0. ~~Study your notes and the textbook (Sects. 3.1–3.3, 4.1–4.3, 4.5–4.6).~~

1. (a) Give the definition of the determinant $\det A$ of an $n \times n$ matrix A, where $n \geq 1$.

   (b) Show that if $A = (a_{ij})_{n\times n}$ is an $n \times n$ matrix with integer entries a_{ij} then $\det A$ is also an integer.

2. (a) Explain what happens to $\det A$ when an elementary row (column) operation is applied to A.

   (b) Evaluate $\det E$ if E is an elementary matrix.

   (c) Suppose that B is a matrix which is row equivalent to A and $\det A = 2$. Find (if possible) $\det B$.

3. Evaluate $\det A$ if A is the following matrix

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -3 & 2 & -5 & 13 \\ 1 & -2 & 10 & 4 \\ -2 & 9 & -8 & 25 \end{bmatrix}; \quad \text{(b)}\quad A = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ .. & .. & \dots & .. & .. \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{n\times n}, \ n = 2, 3, 4, \dots.$$

4. Evaluate $\det A$ if (a) $A = \begin{bmatrix} 1 & n & \dots & n & n \\ n & 2 & \dots & n & n \\ .. & .. & \dots & .. & .. \\ n & n & \dots & n-1 & n \\ n & n & \dots & n & n \end{bmatrix}_{n\times n}$; (b) $A = \begin{bmatrix} b & a & \dots & a & a \\ a & b & \dots & a & a \\ .. & .. & \dots & .. & .. \\ a & a & \dots & b & a \\ a & a & \dots & a & b \end{bmatrix}_{n\times n}$.

5. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ be columns of an $n \times n$ matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \dots\ \mathbf{a}_n]$ and $\det A = 2$. Evaluate the following determinants

   (a) $\det[\mathbf{a}_n\ \mathbf{a}_1\ \mathbf{a}_2\ \dots\ \mathbf{a}_{n-1}]$;

   (b) $\det[\mathbf{a}_1\ 2\mathbf{a}_2\ \dots\ (n-1)\mathbf{a}_{n-1}\ n\mathbf{a}_n]$;

   (c) $\det[\mathbf{a}_1\ \mathbf{a}_1 + \mathbf{a}_2\ \mathbf{a}_2 + \mathbf{a}_3\ \dots\ \mathbf{a}_{n-1} + \mathbf{a}_n]$;

   (d) $\det[\mathbf{a}_1 - \mathbf{a}_2\ \mathbf{a}_2 - \mathbf{a}_3\ \dots\ \mathbf{a}_n - \mathbf{a}_1]$.

6. Find (if possible) $\det 2A$, $\det(-B)$, $\det(A+B)$, $\det(ABA^2B^2)$, ~~$\det(\operatorname{adj} A)$, $\det(\operatorname{adj} B)$~~ if A, B are matrices of size $n \times n$ with $\det A = 1$ and $\det B = 2$.

7. Show that

   (a) An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

   (b) If A, B are $n \times n$ matrices, then $\det AB = \det A \det B$.

8. (a) Write down formulas for the ~~adjoint adj A,~~ and the inverse A^{-1} of a matrix A.

   ~~(b) Explain and prove the false expansion formula and use this formula to show that $\operatorname{adj} A \cdot A = A \cdot \operatorname{adj} A = \det A \cdot I_n$.~~

9. Find (if possible) the ~~adjoint adj~~ A if (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$; (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

10. Explain Cramer's rule and, using this rule (if possible), solve SLE $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = [1, 0, 1]^T$.

11. Give the definition of a *vector space*. Check this definition for $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, $\mathbb{P}_n$, $\mathbb{P}$, $C[a,b]$.

12. (a) Give the definition of a subspace of a vector space V.

    (b) Show that if $v_1,\ldots,v_k$ are vectors in a vector space V then $\text{Span}(v_1,\ldots,v_k)$ is a subspace of V.

13. (a) Is $S=\{(x_1,x_2,x_3,x_4)^T \mid x_1+x_2=x_3+x_4\}$ a subspace of $\mathbb{R}^4$? If so, find a basis and $\dim S$.

    (b) Same problem for the subset $S=\{(x_1,x_2,x_3,x_4)^T \mid x_1+x_2+3=x_3+x_4\}$ of $\mathbb{R}^4$.

    (c) Same problem for $S=\{(x_1,x_2,x_3,x_4)^T \mid x_1+x_2=x_3+x_4, x_1+x_3=x_2+x_4\}$.

14. (a) Is $S=\{p(t)\mid p(0)=0\}$ a subspace of $\mathbb{P}_4$? If so, find a basis and $\dim S$.

    (b) Same problem for the subset $S=\{p(t)\mid p(1)=p(2)\}$ of $\mathbb{P}_3$.

    (c) Same problem for the subset $S=\{p(t)\mid \text{all coefficients of } p(t) \text{ are integers}\}$ of $\mathbb{P}_5$.

15. Explain how to determine whether

    (a) n vectors $v_1,\ldots,v_n\in\mathbb{R}^m$ span $\mathbb{R}^m$;

    (b) n vectors $v_1,\ldots,v_n\in\mathbb{R}^m$ are linearly independent;

    (c) m vectors $v_1,\ldots,v_m\in\mathbb{R}^m$ are linearly independent.

    Give examples of your choice.

16. (a) Give the definition of a *basis* and the *dimension* $\dim V$ of a vector space V.

    (b) Show that if V is a vector space and $(b_1,\ldots,b_n)$ is a basis for V then any m vectors, where $m>n$, are linearly dependent in V.

17. Suppose that B is a matrix in row echelon form.

    (a) Show that pivot columns of B form a basis for Col$B$.

    (b) Show that nonzero rows of B form a basis for Row$B$.

18. Suppose that V is a vector space and $\dim V=n$. Prove that

    (a) If $v_1,\ldots,v_n$ are linearly independent vectors, then $(v_1,\ldots,v_n)$ is a basis for V.

    (b) If $\text{Span}(v_1,\ldots,v_n)=V$, then $(v_1,\ldots,v_n)$ is a basis for V.

19. Let $v_1=(1,0,1,2)^T$, $v_2=(2,1,1,1)^T$, $v_3=(1,0,1,0)^T$, $v_4=(4,1,3,5)^T$.

    (a) Extend (if possible) (v_1,v_2,v_3) to a basis of $\mathbb{R}^4$.

    (b) Are v_1,v_2,v_4 linearly independent?

    (c) Is $\text{Span}(v_1,v_2,v_3,v_4)=\mathbb{R}^4$?

20. Let A be an $m\times n$ matrix. Show that

    (a) ~~$\dim \text{Col}A=\dim \text{Row}A$.~~

    (b) $\text{rank}A+\dim \text{Nul}A=n$.

    (c) SLE $Ax=b$ has a solution if and only if $b\in \text{Col}A$.

    (d) $\text{rank}A=\text{rank}A^T$.

21. Find rank$A$, bases and dimensions of Col$A$, ~~Row$A$,~~ Nul$A$ if

$$\text{(a)}\quad A=\begin{bmatrix}1&2&3&3\\0&3&3&3\end{bmatrix};\quad \text{(b)}\quad A=\begin{bmatrix}1&2&0&1\\2&4&0&2\\4&3&1&3\end{bmatrix}.$$

Math 225 / Problems for Review 3

~~0. Study your notes and the textbook (Sects. 5.1-5.3, 6.1-6.3, 6.5).~~

1. Give definitions of eigenvalues, eigenvectors and eigenspaces of an $n \times n$ matrix A. Show that
(a) λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$;
(b) v is an eigenvector of A, corresponding to an eigenvalue λ, if and only if $v \neq \mathbf{0}$ and $v \in \text{Nul}(A - \lambda I_n)$;
(c) The eigenspace $E(\lambda)$ of A, corresponding to an eigenvalue λ, is $E(\lambda) = \text{Nul}(A - \lambda I_n)$.

2. Let $A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$. (a) Which of $1, 2, -1$ are eigenvalues of A?
(b) Which of $(1,1,1,1)^T$, $(2,1,2,1)^T$, $(0,0,0,0)^T$ are eigenvectors of A?

3. (a) Suppose A is a 3×3 matrix whose eigenvalues are $-7, 1, 2$. Find (if possible) eigenvalues of $A + 5I_3$.
(b) Let $A^2 = 0$, where A is an $n \times n$ matrix. Show that if λ is an eigenvalue of A, then $\lambda = 0$.

4. Prove that an $n \times n$ matrix A is singular (that is, $\det A = 0$) if and only if 0 is an eigenvalue of A.

5. Diagonalize A (if possible), that is, find a representation of the form $A = PDP^{-1}$, where D is diagonal, for
(a) $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$; (b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; (c) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; (d) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

6. Let A be a matrix of Problem 5. Evaluate the product $A^{101}v$, where $v = (-2, 2, 2)^T$, and compute products $A^{101}e_1$, $A^{101}e_2$, $A^{101}e_3$, A^{101}.

7. (a) State a condition that guarantees that an $n \times n$ matrix A is diagonalizable.
(b) Give an example of an $n \times n$ matrix which is not diagonalizable. Prove your answer.

8. Let x, y be vectors in $\mathbb{R}^n$. Show that ~~(a) (Cauchy-Schwartz inequality) $|x \cdot y| \leq \|x\| \cdot \|y\|$.~~
(b) (Pythagorean law) If $x \perp y$ (x and y are orthogonal) then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

~~9. Let H be a subspace of $\mathbb{R}^n$. State the definition of the orthogonal complement $H^\perp$ of H and show that $H^\perp$ is also a subspace, $H \cap H^\perp = \{\mathbf{0}\}$ and $\dim H + \dim H^\perp = n$.~~

~~10. (a) Explain the formulas $\text{Col}(A)^\perp = \text{Nul}(A^T)$, $\text{Col}(A^T)^\perp = \text{Nul}(A)$.~~
~~(b) Find a basis and the dimension of the orthogonal complement $H^\perp$ of H if H is the subspace of $\mathbb{R}^4$ given by $H = \text{Span}((1, -2, 0, 3)^T, (0, 1, 2, 1)^T)$.~~

11. (a) Show that if $a, b \in \mathbb{R}^n$, $a \neq \mathbf{0}$, then the (orthogonal) projection $\text{proj}_a b$ of b onto a is equal to $\frac{a \cdot b}{a \cdot a} a$.
(b) Find the orthogonal projection of $(1, 2, 2, 2)^T$ onto $(3, 2, 1, 2)^T$.
(c) Find the orthogonal projection of $e_1 - e_2 + e_3$ onto $e_1 + e_5$.

12. (a) Give definitions of orthogonal ~~and orthonormal~~ sets of vectors in $\mathbb{R}^n$.
(b) Prove that if $(u_1, \ldots, u_k)$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then vectors $u_1, \ldots, u_k$ are linearly independent.

13. Show that if $W = \text{Span}(u_1, \ldots, u_k)$ is a subspace of $\mathbb{R}^n$, $y \in \mathbb{R}^n$ and $(u_1, \ldots, u_k)$ is an orthogonal basis of W, then the vector $p = \text{proj}_W(y) = \sum_{i=1}^k \frac{y \cdot u_i}{u_i \cdot u_i} u_i$ has the following properties
(a) p is the orthogonal projection of y onto W, ~~that is, $y = p + z$, where $z \in W^\perp$;~~ (b) the length $\|y - w\|$, where $w \in W$, is minimal when $w = p$.

14. Compute $\text{proj}_W(y)$ and $\min \|y - w\|$ over all $w \in W$ if (a) $y = (1, 1, 2, 3)^T$ and $W = \text{Span}(v_1, v_2, v_3)$, $u_1 = (0, 1, 0, 1)^T$, $u_2 = (1, 0, -1, 0)^T$, $u_3 = (1, 0, 1, 0)^T$; (b) $y = (1, 1, 2)^T$ and $W = \text{Nul}([1, 1, 1])$.

~~15. Give the definition of an orthogonal matrix Q of size $n \times n$. Prove that $(u_1, \ldots, u_n)$ is an orthonormal basis for R^n if and only if $U = [u_1 \ldots u_n]$ is an orthogonal matrix.~~

16. State and prove the theorem on least squares solutions of a SLE $A\mathbf{x} = \mathbf{b}$.

17. Find a least squares solution to the the folowing systems of linear equations:

(a) $\begin{aligned} x_1 + x_2 &= 4 \\ x_1 &= 1 \\ x_2 &= 1 \end{aligned}$; (b) $\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \\ x_1 + 2x_2 &= 1 \end{aligned}$.

~~18. Find a best least squares fit by a linear function to the following data:~~

~~(a)~~

x	0	1	2
y	1	2	4

; ~~(b)~~

x	-1	0	1	2
y	0	1	0	0

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