Application to a nonlinear elliptic problem

Note Title 3/3/2009

Consider: $\Omega \subset \mathbb{R}^n$ bounded and open

$I : W^{1,q}(\Omega) \longrightarrow \mathbb{R}$, $1 < q < \infty$

$$I[u] = \int_\Omega L(Du(x), u(x), x)\, dx$$

where
$$L : \mathbb{R}^n \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$$
satisfies Caratheodory conditions:
(i) $L(\cdot, \cdot, x)$ is continuous for almost all $x \in \Omega$
(ii) $L(p, z, \cdot)$ is measurable for all $(p, z) \in \mathbb{R}^n \times \mathbb{R}$
and:

$$(1) \begin{cases} \exists\, C_1 > 0 \text{ and } g_1(x) \in L^1(\Omega) \text{ such that:} \\ |L(p, z, x)| \le C_1(|p|^q + |z|^q) + g_1(x) \\ \text{for all } (p, z) \in \mathbb{R}^n \times \mathbb{R} \text{ and almost all } x \in \Omega. \end{cases}$$

<u>Example</u> If

$$L(p_1, p_2, \dots, p_n, z, x) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) p_i p_j + \frac{1}{2} c(x) z^2 + F(x) z$$

then (i), (ii) and (1) hold provided $a_{ij}, c \in L^\infty(\Omega)$ and $F \in L^2(\Omega)$.
(Use $q = 2$ and $|F(x) z| \le \frac{1}{2} F^2(x) + \frac{1}{2} z^2$)

Lemma 1 If (i) and (ii) are satisfied then, for any measurable functions $q_1, q_2, \dots, q_n, v : \Omega \to \mathbb{R}$, we have

$$L(q_1(\cdot), q_2(\cdot), \dots, q_n(\cdot), v(\cdot), \cdot)$$

is measurable. If in addition (l) is also satisfied then I is well defined.

Proof: Since $q_1, q_2, \dots, q_n, v$ are measurable there exists the sequences of simple functions

$$\{q_1^k\}_{k\in\mathbb{N}}, \{q_2^k\}_{k\in\mathbb{N}}, \dots, \{q_n^k\}_{k\in\mathbb{N}}, \{v^k\}_{k\in\mathbb{N}}$$

such that
$$q_i^k \xrightarrow{a.e} q_i \quad \forall i \in \{1, 2, \dots, n\}$$
$$v^k \xrightarrow{a.e} v$$

Then by (i) we have

$$L^k(x) = L(q_1^k(x), q_2^k(x), \dots, q_n^k(x), v^k(x), x) \xrightarrow{a.e.} L(q_1(x), q_2(x), \dots, q_n(x), v(x), x)$$

Now fix $k \in \mathbb{N}$. Since $q_1^k, q_2^k, \ldots, q_n^k, U^k$ are simple functions there exists $(p_j, z_j) \in \mathbb{R}^n \times \mathbb{R}$ $j \in \{1, 2, \ldots, N\}$ such that

$$\left(q_1^k(x), q_2^k(x), \ldots, q_n^k(x), U^k(x)\right) = \sum_{j=1}^{N} (p_j, z_j)\, \chi_j(x)$$

where χ_j is the characteristic function of a measurable set $\Omega_j \subseteq \Omega$ with

$$\bigcup_{j=1}^{N} \Omega_j = \Omega .$$

Then $L^k \Big|_{\Omega_j} = L(p_j, z_j, \cdot)$ is measurable for all j hence L^k is measurable.

Finally, L is measurable as the limit a.e. of a sequence of measurable functions L^k.

For $U \in W^{1,2}(\Omega)$ we have

$DU = \left(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \ldots, \frac{\partial U}{\partial x_n}\right) \in L^2(\Omega) \times L^2(\Omega) \times \ldots \times L^2(\Omega)$

$U \in L^2(\Omega)$

and by the above argument

$L(DU(\cdot), U(\cdot), \cdot)$ is measurable

In addition, from (1) we have a.e in Ω

$$|L(DU(\cdot), U(\cdot), \cdot)| \le C_1\Big(\underbrace{|DU(\cdot)|^2}_{L^1(\Omega)} + \underbrace{|U(\cdot)|^2}_{L^1(\Omega)}\Big) + \underbrace{g_1(\cdot)}_{\in L^1(\Omega)}$$

So $L(DU(\cdot), U(\cdot), \cdot)$ is integrable on Ω and $|I[U]| < +\infty$. Lemma is finished.

Second assumption is on derivatives of L and insures Gâteaux (even Fréchet) differentiability of I.

(2) For almost all $x \in \Omega$, $L(p_1, p_2, \dots, p_n, z, x)$ has partial derivatives $\frac{\partial L}{\partial p_i}(p, z, x)$, $i = \overline{1,n}$ and $\frac{\partial L}{\partial z}(p, z, x)$ at all $(p, z) \in \mathbb{R}^n \times \mathbb{R}$ which are continuous in (p, z) and satisfy for all $(p, z) \in \mathbb{R}^n \times \mathbb{R}$ and $i = 1, 2, \dots, n$:

$$\left| \frac{\partial L}{\partial p_i}(p, z, x) \right| \leq C_2 \left(|p|^{q-1} + |z|^{q-1} \right) + g_2(x)$$

$$\left| \frac{\partial L}{\partial z}(p, z, x) \right| \leq C_2 \left(|p|^{q-1} + |z|^{q-1} \right) + g_2(x)$$

where $C_2 > 0$ and $g_2 \in L^{q'}(\Omega)$, $\frac{1}{q'} + \frac{1}{q} = 1$, are fixed.

<u>Lemma 2</u> If (i), (ii), (1) and (2) are satisfied then for any $u \in W^{1,q}(\Omega)$ and $v \in W^{1,q}(\Omega)$

$$dI(u)[v] = \lim_{\varepsilon \to 0} \frac{I(u + \varepsilon v) - I(u)}{\varepsilon} \text{ exists and}$$

$$dI(u)[v] = \int_\Omega \sum_{i=1}^{n} \frac{\partial L}{\partial p_i}(\nabla u(x), u(x), x) \frac{\partial v}{\partial x_i}(x) + \frac{\partial L}{\partial z}(\nabla u(x), u, x) v(x) \, dx$$

Proof Fix $u, v \in W^{1,q}(\Omega)$

$dI(u)[v] =$

$$= \lim_{\varepsilon \to 0} \int_\Omega \frac{L((Du+\varepsilon Dv)(x), (u+\varepsilon v)(x), x) - L(Du(x), u(x), x)}{\varepsilon} \, dx$$

Denote the integrand above by $L^\varepsilon(x)$. By (2) L is (Fréchet) differentiable in the first $n+1$ variables for almost all x, hence:

$$L^\varepsilon(x) \xrightarrow{\varepsilon \to 0} \sum_{i=1}^n \frac{\partial L}{\partial p_i}(Du(x), u(x), x) \frac{\partial v}{\partial x_i}(x)$$

$$+ \frac{\partial L}{\partial z}(Du(x), u(x), x) \, v(x) \quad \text{a.e. in } \Omega$$

Also the mean value theorem and the bounds in (2) we have for almost all $x \in \Omega$ and $0 \le |\tilde{\varepsilon}(x)| \le \varepsilon \le 1$:

$$|L^\varepsilon(x)| \le \sum_{i=1}^n \left| \frac{\partial L}{\partial p_i}(Du + \tilde{\varepsilon}(x) Dv, u + \tilde{\varepsilon} v, x) \right| \left| \frac{\partial v}{\partial x_i}(x) \right|$$

$$+ \left| \frac{\partial L}{\partial z}(Du + \tilde{\varepsilon} Dv, u + \tilde{\varepsilon} v, x) \right| |v(x)|$$

$$\le \sum_{i=1}^n \left[C_2 \left(|Du(x) + \tilde{\varepsilon} Dv(x)|^{q-1} + |u(x) + \tilde{\varepsilon}(x) v(x)|^{q-1} \right) + g_2(x) \right] \left| \frac{\partial v}{\partial x_i}(x) \right|$$

$$+ \left[C_2 \left(|Du(x) + \tilde{\varepsilon} Dv(x)|^{q-1} + |u(x) + \tilde{\varepsilon} v(x)|^{q-1} \right) + g_2(x) \right] |v(x)|$$

$$\le \sum_{i=1}^n \left[C_2 2^{q-2} \left(|Du(x)|^{q-1} + |Dv(x)|^{q-1} + |u(x)|^{q-1} + |v(x)|^{q-1} \right) + g_2(x) \right] \left| \frac{\partial v}{\partial x_i}(x) \right|$$

$$+\underbrace{\left[C_2 2^{\varepsilon-2}\left(|Du(x)|^{\varepsilon-1}+|Dv(x)|^{\varepsilon-1}+|u(x)|^{\varepsilon-1}+|v(x)|^{\varepsilon-1}+g_2(x)\right)\right]}_{\in L^{\varepsilon'}(\Omega)}\underbrace{|v(x)|}_{\in L^{\varepsilon}(\Omega)}$$

$$\leq \tilde{g}_2(x) \in L^1(\Omega)$$

By dominated convergence theorem we can pass to the limit in:

$$dI(u)[v] = \lim_{\varepsilon \to 0} \int_\Omega L^\varepsilon(x)\,dx$$

$$= \int_\Omega \sum_{i=1}^{n} \frac{\partial L}{\partial p_i}(Du,u,x)\frac{\partial v}{\partial x_i}(x) + \frac{\partial L}{\partial z}(Du,u,x)\,v(x)\,dx.$$

<u>Remark 1</u> From formula above and continuity of $\frac{\partial L}{\partial p_i}(p,z,x)$, $\frac{\partial L}{\partial z}(p,z,x)$ in (p,z) a.e $x \in \Omega$ we deduce that $dI(u)$ is continuous in u hence I is Fréchet differentiable.

Lemma 2 can be proven assuming only that for a.e $x \in \Omega$, $L(p,z,x)$ has directional derivatives in any direction (p,z) which depends linearly on the direction. In this case I remains Gâteaux differentiable but it may be not Fréchet differentiable.

Theorem 1 Fix $f \in L^q(\partial\Omega)$ such that

$$\mathcal{A} = \{ v \in W^{1,q}(\Omega) \mid \text{trace } v = f \} \neq \emptyset .$$

If (i), (ii) (1) and (2) are satisfied and v is a minimizer (maximizer) of I relative to $\mathcal{A}$ then v is a weak solution of the boundary value problem:

$$(BVP) \begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{i=1}^{n} \frac{\partial L}{\partial p_i}(Dv, v, x) \right) + \frac{\partial L}{\partial z}(Dv, v, x) = 0 \\ v|_{\partial\Omega} = f \end{cases}$$

Remark 2 For the example on page 1 the BVP is:

$$(\widetilde{BVP}) \begin{cases} -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + c(x) v(x) + F(x) = 0 \quad x \in \Omega \\ v|_{\partial\Omega} = f \end{cases}$$

Given Lemma 1 & 2 (BVP) should be a direct consequence of Euler-Lagrange equations for constrained minimizers, see Lecture 7. However the constrained $v \in \mathcal{A}$ cannot be written in

the form $G(u)=0$ for $G: X \to \mathbb{R}$ (or $\mathbb{R}^n$).
As a project you are invited to generalize the result in Lecture 7 such that it can be directly applied to this problem.

<u>Proof of Theorem 1</u> Fix $w \in \mathcal{A}$, note that

$$\mathcal{A} = w + W_0^{1,q}(\Omega) = \{ w+u \mid u \in W_0^{1,q}(\Omega) \}$$

Define $\tilde{I} : W_0^{1,q}(\Omega) \to \mathbb{R}$ by

$$\tilde{I}(u) = I(w+u)$$

Lemma 1 implies $\tilde{I}$ is well defined while Lemma 2 guarantees $\tilde{I}$ has Gâteaux differential everywhere:

$$\delta \tilde{I}(u)[v] = \delta I(w+u)[v]$$

$$= \int_\Omega \sum_{i=1}^{n} \frac{\partial L}{\partial p_i}(Dw+Du, w+u, x) \frac{\partial v}{\partial x_i}(x) + \frac{\partial L}{\partial z}(Dw+Du, w+u, x) v \, dx$$

for all $v \in W_0^{1,q}(U)$
Moreover if u is a minimizer of $\tilde{I}$ relative

to $\mathcal{A}$ then $\tilde{v} = u - w$ is a global minimizer of $\tilde{I}$ on $W_0^{1,q}(\Omega)$. By first theorem in Lecture 7 we must have:

$$\delta \tilde{I}(\tilde{v})[v] = 0 \quad \forall v \in W_0^{1,q}(\Omega)$$

Therefore:

$$(*) \quad \int_\Omega \sum_{i=1}^{n} \frac{\partial L}{\partial p_i}(Du, u, x)\frac{\partial v}{\partial x_i} + \frac{\partial L}{\partial z}(Du, u, x)\, v \, dx = 0 \quad \forall v \in W_0^{1,q}(\Omega)$$

Hence u is a weak sln of BVP (which require $(*)$ for $v \in C_c^\infty(\Omega)$ only). q.e.d.

To guarantee the existence of a minima for I coercivity and convexity assumptions are made:

(3) Coercivity: Assume $I[u] \geq \alpha \|Du\|^q_{L^q(\Omega)} - \beta$ for some constants $\alpha > 0$, $\beta \in \mathbb{R}$ and any $u \in \mathcal{A}$

Now (3) implies coercivity of I on $\mathcal{A}$ i.e.

$$\lim_{\|u\|_{W^{1,q}} \to \infty,\ u \in \mathcal{A}} I[u] = +\infty$$

Indeed $u \in \mathcal{A} \Rightarrow u - w \in W_0^{1,2}(\Omega)$ for some fixed $w \in \mathcal{A}$. Since Ω is bounded by Poincaré Inequality:

$$\|u-w\|_{L^2} \le C_p \|Du - Dw\|_{L^2}$$

$$\Rightarrow \|u\|_{L^2} \le C_p \|Du\|_{L^2} + \max\{1, C_p\} \|w\|_{W^{1,2}(\Omega)}$$

$$\Rightarrow \|u\|_{W^{1,2}(\Omega)} \le (C_p+1)\left(\|Du\|_{L^2} + \|w\|_{W^{1,2}(\Omega)}\right)$$

Since w is fixed, $\|u\|_{W^{1,2}(\Omega)} \to \infty$ $u \in \mathcal{A}$ implies

$$\|Du\|_{L^2(\Omega)} \to \infty$$

so by (3) we get

$$\lim_{\substack{\|u\|_{W^{1,2}(\Omega)} \to \infty \\ u \in \mathcal{A}}} I[u] = +\infty.$$

i.e. coercivity of I on $\mathcal{A}$.

<u>Remark 3</u> Note that we used Ω bounded (in Poincaré Ineq) to show (3) implies coercivity.

<u>Lemma 3</u> If $\exists\, C_3 > 0$, $0 \le r < q$,
$g_3(x) \in L^{\alpha}(\Omega)$ with (recall $\Omega \subseteq \mathbb{R}^n$)

$$\alpha = 1 \quad \text{if } r = 0$$
$$\frac{1}{r}\left(\frac{1}{r} + \frac{1}{n} - \frac{1}{q}\right)^{-1} \le \alpha \le \infty \quad \text{if } q < n$$
$$1 < \alpha \le \infty \quad \text{if } q = n$$
$$1 \le \alpha \le \infty \quad \text{if } q > n$$

and $\tilde{g}_3(x) \in L^1(\Omega)$ such that, for almost all $x \in \Omega$:

$$(3') \qquad L(p, z, x) \geq C_3 |p|^q - g_3(x)|z|^r - \tilde{g}_3(x)$$

then (3) is satisfied and I is coercive <u>on $\mathcal{A}$</u>.

<u>Proof</u> For any $v \in W^{1,q}(\Omega) \cap \mathcal{A}$ we have:

$$L(Dv(x), v(x), x) \geq C_3 |Dv(x)|^q - g_3(x)|v(x)|^r - \tilde{g}_3(x)$$
a.e. on $x \in \Omega$

hence

$$I[v] = \int_\Omega L(Dv, v, x)\,dx \geq C_3 \|Dv\|^q_{L^q} - \|\tilde{g}_3\|_{L^1} - \int_\Omega g_3(x)|v(x)|^r dx$$

For the last integral term use generalised Hölder:

$$\int_\Omega g_3(x)\cdot 1\cdot |U(x)|^r\,dx \le \|g_3\|_{L^\alpha}(\operatorname{meas}\Omega)^{1/\beta}\,\|U\|^r_{L^{\gamma r}}$$

where $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=1 \quad 1\le\alpha,\beta,\gamma\le\infty$

the range of α in the hypothesis and $1\le\beta\le\infty$ allows us to choose:

$$q\le r\gamma\le\frac{nq}{n-q} \quad \text{if } n>q.$$

$$q\le r\gamma<\infty \quad \text{if } n=q \text{ and}$$

$$q\le r\gamma\le\infty \quad \text{if } n<q.$$

Hence by Sobolev Imbedding:

$$\|U\|_{L^{\gamma r}}\le C\,\|U\|_{W^{1,q}(\Omega)}$$

As before, from Ω bounded and $U\in\mathcal{A}\iff U\in w+W_0^{1,q}(\Omega)$ with $w\in W^{1,q}(\Omega)$ fixed we get via Poincaré Ineq:

$$\|U\|_{W^{1,q}}\le (C_p+1)\left(\|DU\|_{L^q}+\|w\|_{W^{1,q}}\right)$$

In conclusion for $w\in\mathcal{A}$ fixed and $r<q$:

$$I[U]\ge C_3\|DU\|^q_{L^q}-C\|DU\|^r_{L^q}-C\|w\|^r_{W^{1,q}(\Omega)}-\|\tilde g_3\|_{L^1}$$

$$\ge \tilde C_3\|DU\|^q_{L^q}-C\|w\|^r_{W^{1,q}(\Omega)}-\|\tilde g_3\|_{L^1}$$

and the Lemma is finished.

(4) Convexity: Assume for almost all $x \in \Omega$ and all $z \in \mathbb{R}$ we have
$$p \longmapsto L(p, z, x) \text{ is convex (in } p).$$

Remark 4 (4) does not imply convexity of I but it is sufficient to show its weak lower semicontinuity on $\mathcal{A}$ based on continuity of $D_p L$ in z.

Lemma 4: Assume (i), (ii), (1), (2) and (4) hold then $I : \mathcal{A} \to \mathbb{R}$ is weakly lower semicontinuous.

Proof Let $u_k \rightharpoonup u$ in $W^{1,2}(\Omega)$, $\{u_k\} \subseteq \mathcal{A}$.
We need to show:
$$I[u] \leq \liminf I[u_k] \overset{\text{notation}}{=} \ell$$
Fix $w \in \mathcal{A} \Rightarrow \{u_k - w\} \subseteq W_0^{1,2}(\Omega)$ and
$$u_k - w \rightharpoonup u - w \text{ in } W_0^{1,2}(\Omega)$$
because any linear, continuous functional on $W_0^{1,2}(\Omega)$ can be extended via Hahn-Banach Th to a continuous, linear functional on $W^{1,2}(\Omega)$. In particular $u \in \mathcal{A}$.

Now we choose a subsequence $\{u_{k'}\}$ of $\{u_k\}$ such that $\ell = \liminf I[u_k] = \lim\limits_{k' \to \infty} I[u_{k'}]$ and a subsequence $\{v_{k''}\}$ of $\{v_{k'}\}$ such that

$$v_{k''} \xrightarrow{L^2} v \quad \text{and} \quad v_{k''}(x) \to v(x) \text{ a.e in } \Omega$$

Note that $\{v_{k''}\}$ exists because $v_{k'} - w \xrightarrow{W_0^{1,2}(\Omega)} v - w$ and $W_0^{1,2}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^2(\Omega)$ since Ω is bounded.

Redenote $\{v_{k''}\}$ by $\{v_k\}$. All in all we have:

$$v_k \xrightarrow{W^{1,2}(\Omega)} v$$
$$v_k \xrightarrow{L^2(\Omega)} v, \quad v_k \xrightarrow{a.e.} v$$
$$\lim_{k\to\infty} I[v_k] = l.$$

Fix $x \in \Omega$ such that $v_k(x) \longrightarrow v(x)$. $D_p L(p, \cdot, x)$ is continuous $\forall p \in \mathbb{R}^n$, see (2), and $L(\cdot, z, x)$ is convex $\forall z \in \mathbb{R}$, see (4). Then by Theorem 1 and Remark 1 in Lecture 7:

$$L(Dv_k(x), v_k(x), x) \geq L(Dv(x), v_k(x), x) + D_p L(Dv(x), v_k(x), x)[Dv_k(x) - Dv(x)]$$

Since for almost all $x \in \Omega$ the above properties are valid, the inequality above hold a.e. $x \in \Omega$. Integrating over Ω we get

(*) $I[u_k] \geq \int_\Omega L(Du(x), u_k(x), x)\,dx$

$$+ \int_\Omega D_pL(Du(x), u(x), x)\,[Du_k(x) - Du(x)]\,dx$$

$$+ \int_\Omega [D_pL(Du(x), u_k(x), x) - D_pL(Du(x), u(x), x)]\,[Du_k(x) - Du(x)]\,dx$$

From $u_k(x) \xrightarrow{a.e.} u(x)$ and

$L(p, \cdot, x)$ continuous a.e. $x \in \Omega$, $\forall p \in \mathbb{R}^n$

$D_pL(p, \cdot, x)$ continuous a.e. $x \in \Omega$, $\forall p \in \mathbb{R}^n$

we get

$$L(Du(x), u_k(x), x) \xrightarrow{a.e.} L(Du(x), u(x), x)$$

$$D_pL(Du(x), u_k(x), x) \xrightarrow{a.e.} L(Du(x), u(x), x)$$

and via dominated convergence theorem:

$$L(Du(\cdot), u_k(\cdot), \cdot) \xrightarrow{L^1(\Omega)} L(Du(\cdot), u(\cdot), \cdot)$$

$$D_pL(Du(\cdot), u_k(\cdot), \cdot) \xrightarrow{L^{q'}(\Omega)} D_pL(Du(\cdot), u(\cdot), \cdot)$$

Moreover $u_k \xrightarrow{W^{1,q}} u \Rightarrow Du_k \xrightarrow{L^q} Du$ hence

$$\int_\Omega \underbrace{D_pL(Du(x), u(x), x)}_{\text{fixed element of } (L^{q'}) = (L^q)^*}\,[Du_k(x) - Du(x)]\,dx \xrightarrow{k\to\infty} 0$$

and $Du_k \xrightarrow{L^2} Du \Rightarrow \exists M > 0$ such that

$$\| Du_k - Du \|_{L^2} \leq M \quad \forall k \in \mathbb{N}$$

hence by Hölder

$$\int_\Omega \left[D_p L(Du(x), u_k(x), x) - D_p L(Du(x), u(x), x) \right] \left[Du_k(x) - Du(x) \right] dx$$

$$\leq \| D_p L(Du(\cdot), u_k(\cdot), \cdot) - D_p L(Du(\cdot), u(\cdot), \cdot) \|_{L^{2'}} \| Du_k - Du \|_{L^2}$$

$\longrightarrow 0$ as a product of a seq convergent to zero and a bounded sequence.

In conclusion, by passing to the limit $k \to \infty$ in $(*)$ we get

$$l = \lim_{k \to \infty} I[u_k] \geq \int_\Omega L(Du(x), u(x), x)\, dx = I[u]$$

and the Lemma is finished.

Remark 5 (i) The proof of Lemma 4 also shows that

$$v_k \rightharpoonup v, \ \{v_k\} \subseteq \mathcal{A} \implies v \in \mathcal{A}$$

(ii) If Ω satisfies the cone property, then the same proof and

$$W^{1,q}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^q(\Omega)$$

shows I weakly lower semicontinuous on $W^{1,q}(\Omega)$.

(iii) I weakly lower semicontinuous on $\mathcal{A}$ implies for $w \in \mathcal{A}$ fixed:

$$\tilde{I}[\cdot] = I[\cdot - w]$$

is weakly lower semicontinuous on $W_0^{1,q}(\Omega)$.

Theorem 2 Assume (i), (ii), (1)–(4) are all satisfied then I has at least one global minimum on $\mathcal{A}$ provided $\mathcal{A}$ is nonempty.

Proof Use Lemmas 1 and 4 and the

proof of Theorem 2 in Lecture 7 (general theorem for existence of minimizers) where the argument on the limit being in the admissible set is replaced by Remark 5(i).

Alternatively one can fix $w \in \mathcal{A}$ and
$$\tilde{I} : W_0^{1,q}(\Omega) \to \mathbb{R} \qquad \tilde{I}[\cdot] = I[\cdot - w]$$
Apply Theorem 2 in Lecture 7 to $\tilde{I}$ (with no constraints, i.e. $G : W_0^{1,q}(\Omega) \to \mathbb{R}$ $G(\cdot) \equiv 0$).
Then use $\tilde{v}$ is a minimizer of $\tilde{I}$ $\Leftrightarrow$ $\tilde{v}+w$ is a minimizer of I on $\mathcal{A}$. Q.E.D.

<u>Corollary 3</u> If (i), (ii), (1)–(4) are all satisfied then the boundary value problem

$$\begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial p_i}(Du,u,x)\right) + \frac{\partial L}{\partial z}(Du,u,x) = 0 \\ u|_{\partial\Omega} = f \end{cases}$$

has at least one weak solution provided $\exists\, w \in W^{1,q}(\Omega)$ s.t. $w = f$.

<u>Proof</u> This is an immediate consequence of Theorems 2 and 1.

For uniqueness:

(4') Assume that for almost all $x \in \Omega$
$$(p, z) \longmapsto L(p, z, x)$$
is strictly convex in (p, z).

<u>Corollary 4</u> If (i), (ii), (1)–(3) and (4') are satisfied then $I : \mathcal{A} \to \mathbb{R}$ is strictly convex, has a unique minimizer, and the boundary value problem

$$(BVP) \quad -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial p_i}(Du(x), u(x), x)\right) + \frac{\partial L}{\partial z}(Du(x), u(x), x) = 0$$
$$u\big|_{\partial\Omega} = f$$

has a unique weak sln.

<u>Proof</u> Let $u \neq v \in \mathcal{A}$ (note that $\mathcal{A}$ is convex). Fix $t \in (0,1)$. Then from (4'), for almost all $x \in \Omega$

$$L((1-t)Du(x) + tDv(x), (1-t)u(x) + tv(x), x)$$
$$< (1-t)L(Du(x), u(x), x) + t\,L(Dv(x), v(x), x)$$

Integrating over Ω we get

$I((1-t)u+tv) < (1-t)I(u)+tI(v)$ hence I is strictly convex (actually on $W^{1,2}(U)$).

Then from Remark 3 in Lecture 7 we get that the minimizer is unique. Its existence is insured by Theorem 2.

Assume now u is a weak sln of (BVP). By definition of weak sln:

$$\delta I(u)[\varphi] = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

Since (i), (ii), (1) and (2) imply I Fréchet differentiable everywhere, see Remark 1 after Lemma 2, we have that I is continuous and:

$$DI(u)[\varphi] = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

and by density of $C_c^\infty(\Omega)$ in $W_0^{1,2}(\Omega)$ plus continuity of $DI(u)$ we get

$$DI(u)[v] = 0 \quad \forall v \in W_0^{1,2}(\Omega).$$

Moreover, continuity and convexity of I implies via Theorem 1 in Lecture 7 that

$\exists\, h \in (W^{1,2}(\Omega))^*$ such that

$$I(\tilde{v}) \geq I(v) + \langle h, \tilde{v} - v \rangle \quad \forall\, \tilde{v} \in W^{1,2}(\Omega)$$

But for Fréchet differentiable functionals $h = DI(v)$
Hence for any $\tilde{v} \in \mathcal{A}$

$$I(\tilde{v}) \geq I(v) + \langle DI(v), \underbrace{\tilde{v} - v}_{\in W_0^{1,2}(\Omega)} \rangle = I(v)$$

So v is a minimizer of I on $\mathcal{A}$ and by strict convexity v is unique. QED

<u>Example</u> If

$$L(p_1, p_2, \ldots, p_n, z, x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) p_i p_j$$

$$+ \frac{1}{2} c(x) z^2 + F(x) z$$

then the boundary value problem is:

$$(\widetilde{BVP}) \begin{cases} -\sum\limits_{i,j=1} a_{ij} \dfrac{\partial^2 v}{\partial x_i \partial x_j} + c(x) v(x) + F(x) = 0 & x \in \Omega \\ v|_{\partial\Omega} = f \end{cases}$$

Moreover (i), (ii), (1) and (2) are satisfied provided $a_{ij}(x), c(x) \in L^\infty(\Omega)$, $F \in L^2(\Omega)$.
(3') which implies (3), see Lemma 3, is satisfied provided $c(x) \geq 0$ a.e. in Ω and:

(Uniform ellipticity condition) $\exists\, C_1 > 0$ such that for almost all $x \in \Omega$:

$$\sum_{i=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geq C_1 |\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$

Note that the uniform ellipticity condition is equivalent to: for almost all $x \in \Omega$

$$\sum_{i=1}^{n} \frac{\partial^2 L}{\partial p_i \partial p_j}(p, z, x)\, \xi_i \xi_j \geq C |\xi|^2 \quad \forall (p, z) \in \mathbb{R}^n \times \mathbb{R}, \ \forall \xi \in \mathbb{R}^n$$

which implies strict convexity of $p \mapsto L(p, z, x)$ for all z and almost all $x \in \Omega$, hence (4) is satisfied.

<u>Corollary 5</u> If $a_{ij}, c \in L^\infty(\Omega)$, $F \in L^2(\Omega)$ such that $c(x) \geq 0$ a.e. in Ω and $\exists\, C_1 > 0$ such that for almost all $x \in \Omega$:

$$\sum_{i=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geq C |\xi|^2 \quad \forall \xi \in \mathbb{R}^n$$

then $(\widetilde{BVP})$ has exactly one weak solution.

Proof We already checked that (i), (ii), (1)–(4) are all satisfied, so there is at least one solution according to Corollary 3.

Now (4′) is not quite satisfied since

$$L(p, (1-t)z_1 + tz_2, x) = (1-t)\,L(p, z_1, x) + t\,L(p, z_2, x)$$

for $z_1 \neq z_2$ and $x \in \Omega$ for which $c(x) = 0$.

But $I[v]$ remains strictly convex simply because $u \neq v$ in $W^{1,2}(\Omega) \Rightarrow Du \neq Dv$ in $L^2(\Omega)$, so

$$L((1-t)Du + tDv, (1-t)u + tv, x) <$$

$$< (1-t)\,L(Du, u, x) + t\,L(Dv, v, x)$$

by strict convexity of L in the first n variables and convexity in the $n+1^{\text{th}}$.

So the argument in Corollary 4 still applies and the weak sln is unique. QED.