Lecture 1 Math 555

Note Title 1/13/2008

Summary

1. Organisational issues

2. Overview of the material to be covered

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— First day handout

2. Overview of the material to be covered

1. <u>Bifurcation Theory in PDE's</u>

An example from ODE's:

$$\frac{dx}{dt} = F(\lambda, x) \qquad x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}$$

$F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and:

$$F(\lambda, 0) = 0 \quad \forall\ \lambda \in \mathbb{R}$$

We are looking for equilibrium solutions:

(1) $$F(\lambda, x) = 0$$

0 is an equilibrium for all $\lambda \in \mathbb{R}$.

<u>Case I</u> $D_x F(\lambda_0, 0)$ (which is a $n \times n$ matrix) is invertible.

Then, by implicit function theorem, (1) has a unique sln $x \equiv 0$ for $|\lambda - \lambda_0|$ and $|x|$ sufficiently small.

<u>Case II</u> $D_x F(\lambda_0, 0)$ is not invertible, then (1) may have other solutions besides $x = 0$ (which "bifurcate" from $x = 0$ at $\lambda = \lambda_0$):

Assume
$$\ker D_x F(\lambda_0, 0) = \operatorname{span}\{v\}$$
Then any $x \in \mathbb{R}^n$ can be written in a unique way:
$$x = a v + x_1, \quad x_1 \in \operatorname{Range} D_x F(\lambda_0, 0)$$

Similarly:

$$F(\lambda, x) = F_1(\lambda, x)\, v + F_2(\lambda, x)$$

with $F_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and

$$F_2 : \mathbb{R} \times \mathbb{R}^n \to \text{range } D_x F(\lambda_0, 0)$$

(1) becomes:

$$G(\lambda, a, x_1) = F_2(\lambda, a v + x_1) = 0$$

$$F_1(\lambda, a v + x_1) = 0$$

Since $G(\lambda_0, 0, 0) = 0$ and

$$D_{x_1} G(\lambda_0, 0, 0) = D_x F \Big|_{\text{range } D_x F}$$ is

invertible the first eq, via implicit function theorem, gives a C^1 surface of sln's:

$$x_1 = x_1(\lambda, a) \quad \text{with } x_1(\lambda_0, 0) = 0$$

Replacing in the second eq:

$$G_1(\lambda, a) = F_1(\lambda, a v + x_1(\lambda, a)) = 0$$

with $G_1(\lambda_0, 0) = 0$

$$\frac{\partial G_1}{\partial \lambda}(\lambda_0, 0) = \frac{\partial F_1}{\partial \lambda}(\lambda_0, 0) + \cancelto{0}{\nabla_x F_1(\lambda_0, 0) \cdot \left(0, \frac{\partial x_1}{\partial \lambda}\right)}$$

$$= \frac{\partial F_1}{\partial \lambda}(\lambda_0, 0)$$

In general $\frac{\partial F_1}{\partial \lambda}(\lambda_0, 0) \neq 0$ and, via implicit function theorem, we get a unique sln

$$\lambda = \lambda(a), \quad \lambda(0) = \lambda_0$$

Hence (1) has a curve a sln's branches $x=0$ given by $a \in \mathbb{R}$, $|a|$ small:

$$\begin{cases} x = a v + x_1(a, \lambda(a)) \\ \lambda = \lambda(a). \end{cases}$$

Questions: how we generalize such technique to the case when F contains differential operators for example:

$$F(\lambda, x) \longmapsto F(\lambda, u) = [-\Delta + V(x)]u + |u|^2 u + \lambda u$$

$$F : \mathbb{R} \times H^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

We will need first to generalize the implicit function theorem to abstract Banach spaces.

2. Applications of semigroup of operators in PDE's.

Start from a system of ODE's:

$$(2)\quad \begin{cases} \dfrac{dx}{dt} = Ax + F(x), & x \in \mathbb{R}^n \\ x(0) = x_0 \end{cases}$$

A is an $n \times n$ matrix and $F: \mathbb{R}^n \to \mathbb{R}^n$.

Duhamel Principle says that any C^1 solution of:

$$(3)\quad x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} F(x(s))\,ds$$

is also a sln of (2). This can be checked by differentiation.

(3) is used to show asymptotic stability of the zero solution under the hypotheses:

(i) e-values of A have real part negative,

(ii) $F(0) = 0$, $DF(0) = 0$.

In the PDE case A is replaced by a partial differential operator, for example:

$$\frac{du}{dt} = \Delta u + F(u). \qquad u : \mathbb{R}^n \to \mathbb{R}$$

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2}$$

To use (3) we need to make sense of

$$e^{\Delta t}$$

which is the subject of theory of semigroup of operators.

3. Variational methods in PDE's:

An ODE example: $x(t) \in \mathbb{R}^n$

$$\frac{d^2 x}{dt^2} = F(x) = -\frac{dV}{dx}$$

Equilibria are given by critical points of $V(x)$, in particular by its local maxima and minima. Moreover local minima give stable equilibria. Such results can be generalized to Hamiltonian systems and many PDE's have such a structure.