Lecture 10 Math 555

Note Title 3/10/2008

Applications of variational methods to nonlinear Schrödinger equation. Lyapunov functionals and orbital stability.

Recall: Lyapunov Method and stability of solutions of systems of ordinary differential equations

Consider the system of ODE's:

(1) $\dot{x}(t) = F(t, x(t)) \qquad x : I \subseteq \mathbb{R} \to \mathbb{R}^n$

where

1° $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in first variable and locally Lipschitz in the second variable

2° $F(t, 0) = 0$ for all $t \in \mathbb{R}$

Consequently $x(t) \equiv 0$ is a solution.

Def (stability) $x(t) \equiv 0$ is a stable solution of (1) if for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$ $\exists \delta(t_0, \varepsilon)$ such that any continuous solution of (1) with $|x(t_0)| < \delta$ can be extended to $+\infty$ and $|x(t)| < \varepsilon$ for $t_0 \leq t < +\infty$

Theorem (Lyapunov functional method)
Assume $\exists \delta > 0$ and

$$V: B(0,\delta) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$$

such that

(i) V is continuous.

(ii) $V(x) \geq 0$ and $V(x) = 0$ iff $x = 0$

(iii) For any solution $x(t)$, $a < t < b$, of (1) with $|x(t)| < \delta$ for $a < t < b$ we have:

$$V(x(t)) : (a,b) \to \mathbb{R} \text{ is nonincreasing.}$$

Then $x(t) \equiv 0$ is a stable sln of (1).

Remark 3° is sometimes replaced by the stronger condition: V is differentiable on $B(0,\delta)$ and

$$F(t,x) \cdot \nabla V(x) \leq 0 \quad \forall t \in \mathbb{R}, |x| < \delta.$$

Proof of Theorem (sketch) Fix $t_0 \in \mathbb{R}$ assume $\exists \ \varepsilon > 0$ and sequences

$$x_n^0 \longrightarrow 0, \quad t_n > t_0, \quad n \in \mathbb{N}$$

such that for any $n \in \mathbb{N}$ a continuous solution of (1) with initial condition $x(t_0) = x_n^0$ satisfies

$$|x(t_n)| \geq \varepsilon.$$

Pick $0 < \varepsilon_1 < \min\{\varepsilon, \delta\}$. Then $\exists \ n_1 > 0$ such that $|x_n^0| < \varepsilon_1 \quad \forall n \geq n_1$. For each $n \geq n_1$ fix a solution $x_n(t)$ of (1) such that $x_n(t_0) = x_n^0$ and $x_n(t_n) \geq \varepsilon$. Choose

$$\tilde{t}_n = \inf \{ t \mid |x_n(t)| > \varepsilon_1 \}.$$

By continuity of $x_n(t)$ we have

$$\begin{cases} x_n(\tilde{t}_n) = \varepsilon_1 \\ |x_n(t)| < \varepsilon_1 \quad t_0 \leq t < \tilde{t}_n \end{cases}$$

Hence $V(x_n(t)) : [t_0, \tilde{t}_n] \longrightarrow \mathbb{R}$ is well defined and nonincreasing. Moreover

$$0 \leq V(x_n(\tilde{t}_n)) \leq V(x_n(t_0)) = V(x_n^0)$$

$$\xrightarrow{n \to \infty} 0 \xleftarrow{\text{as } n \to \infty}$$

Since $|x_n(\tilde{t}_n)| = \varepsilon_1$ is bounded $\exists\, y \in \mathbb{R}^n$, $|y| = \varepsilon_1$ and $x_{n_k}(\tilde{t}_{n_k}) \longrightarrow y$ as $k \to \infty$.

By continuity of V and the above we have:

$0 = V(y)$, $|y| = \varepsilon_1 > 0$ contradiction with (ii)

So $x(t) \equiv 0$ is stable <u>QED</u>.

Back to nonlinear Schrödinger equations:

Consider

(2) $\quad i\frac{\partial \varphi}{\partial t} = -\Delta\varphi - |\varphi|^{\alpha}\varphi \qquad 0<\alpha<\frac{4}{N}$

$\varphi \in C(\mathbb{R}, H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^N))$.

We saw in lecture 9 that

$$\mathcal{H}(u) = \frac{1}{2}\int|\nabla u|^2 dx - \frac{1}{\alpha+2}\int |u|^{\alpha+2}dx \ , \quad u\in H^1(\mathbb{R}^N)$$

subject to constrained $\|u\|_{L^2}^2 = \mu > 0$ has a unique minimizer $u_\mu \in H^1(\mathbb{R}^N)$ up to translation and rotations.

In addition u_μ satisfies

(3) $\qquad -\Delta u_\mu - |u_\mu|^{\alpha} u_\mu = \lambda u_\mu$ for some $\lambda<0$

hence $\varphi_\lambda = e^{-i\lambda t}\, u_\mu(x)$ is a solution of (2) which is called "ground state".

<u>Theorem</u> $\varphi_\lambda(t.)$ is a "stable" solution of (2) in the sense that:

$\forall t_0 \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta(t_0, \varepsilon_0)$ such that for any solution $\varphi(t)$ of (2) satisfying

$$\|\varphi(t_0) - \varphi_\lambda(t_0)\|_{H^1} < \delta$$

we have

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \inf_{y\in\mathbb{R}^N} \|\varphi(t) - e^{i\theta}\varphi_\lambda(t, \cdot - y)\|_{H^1} < \varepsilon$$

in other words $\exists\ \theta(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}^N$ such that:

$$\sup_{t\in\mathbb{R}} \|\varphi(t) - e^{i\theta(t)}\varphi_\lambda(t, \cdot - y(t))\|_{H^1} < \varepsilon$$

<u>Remark</u> The theorem implies that solutions of (2) starting close to a groundstate remain close to the orbit of the groundstate modulo space translations (orbital stability modulo space translation)

Space translations are necessary:

Consider $U^\varepsilon(x) = e^{i\varepsilon x\cdot v}\, U_\mu(x)$, $v \in \mathbb{R}^N$, $|v| = 1$

and
$$\varphi^\varepsilon(t) = e^{i\varepsilon(x\cdot v - \varepsilon t)}\, e^{-i\lambda t}\, U_\mu(x - 2\varepsilon t v)$$

then $\varphi^\varepsilon(t)$ are slu's of (2) satisfying.
$$\varphi^\varepsilon(0) = U^\varepsilon(x) \xrightarrow[H^1]{} U_\mu(x) \text{ as } \varepsilon \to 0$$

but
$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \| \varphi^\varepsilon(t) - e^{i\theta} \underbrace{e^{-i\lambda t}\, U_\mu(x)}_{\varphi_\lambda(t)} \|_{H^1} = 2\|U_\mu\|_{H^1}$$

Rotations are necessary:

Consider $U^\varepsilon(x) = (1+\varepsilon)^{\frac{1}{\alpha}}\, U_\mu((1+\varepsilon)^{\frac{1}{2}} x)$

and
$$\varphi^\varepsilon(t) = e^{-i\lambda(1+\varepsilon)t}\, (1+\varepsilon)^{\frac{1}{2}}\, U_\mu((1+\varepsilon)^{\frac{1}{2}} x)$$

then $\varphi^\varepsilon(t)$ are slu's of (2) satisfying.
$$\varphi^\varepsilon(0) = U^\varepsilon \xrightarrow{H^1} U_\mu \text{ as } \varepsilon \to 0$$

but
$$\sup_{t\in\mathbb{R}} \inf_{y\in\mathbb{R}^N} \| \varphi^\varepsilon(t) - \varphi_\lambda(t, \cdot - y) \|_{H^1} \geq \| U_\mu \|_{H^1}$$

Before we can prove the theorem we need to prove the following conservation laws:

<u>Lemma</u> (conservation of mass and energy)
Along any $\varphi \in C(\mathbb{R}, H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^N))$ solution of (2) the following are conserved

a) $N(\varphi(t)) = \int_{\mathbb{R}^N} |\varphi(t)|^2 dx \equiv \text{const}$

b) $\mathcal{H}(\varphi(t)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \varphi(t)|^2 - \frac{1}{\alpha+2}\int_{\mathbb{R}^N} |\varphi(t)|^{\alpha+2} dx$

<u>Proof</u> In what follows $\langle v, v\rangle$ denotes the pairing $v \in H^{-1}$, $v \in H^1$ which coincides with the (complex) scalar product in L^2 if $v, v \in L^2$.

a) From (2) we have

$$\langle i\frac{d\varphi}{\partial t}(t), \varphi(t)\rangle = \langle -\Delta\varphi, \varphi\rangle - \langle |\varphi|^{\alpha}\varphi, \varphi\rangle$$

$$-i\langle \frac{d\varphi}{\partial t}, \varphi\rangle = \|\nabla\varphi\|_{L^2}^2 - \|\varphi\|_{L^{\alpha+2}}^{\alpha+2}$$

Taking the complex conjugate:

$$i \langle \varphi, \frac{d\varphi}{\partial t} \rangle = \|\nabla\varphi\|_{L^2}^2 - \|\varphi\|_{L^{\alpha+2}}^{\alpha+2}$$

But $\frac{d}{dt} W(\varphi(t)) = \frac{d}{dt} \langle \varphi(t), \varphi(t) \rangle =$

$$= \langle \frac{d\varphi}{dt}, \varphi \rangle + \langle \varphi, \frac{d\varphi}{dt} \rangle = 0$$

see above calculation

$\Rightarrow W(t) = \text{const.}$

b) Formally

$$\frac{d}{dt} \|\nabla\varphi\|_{L^2}^2 = \frac{d}{dt} \langle \nabla\varphi, \nabla\varphi \rangle = \langle \frac{d\varphi}{dt}, -\Delta\varphi \rangle + \langle -\Delta\varphi, \frac{d\varphi}{dt} \rangle \quad (*)$$

<u>Problem</u> In $(*)$ $\frac{d\varphi}{dt} \in H^{-1}$ and $-\Delta\varphi \in H^{-1}$ so the derivative might not exist.

$$(**) \quad \frac{d}{dt} \|\varphi\|_{L^{\alpha+2}}^{\alpha+2} = \frac{d}{dt} \int_{\mathbb{R}^N} |\varphi(t)|^{\alpha+2} dx = \frac{\alpha+2}{2} \int_{\mathbb{R}^N} |\varphi|^{\alpha-1} \frac{\partial}{\partial t} |\varphi|^2 dx$$

$$= \frac{\alpha+2}{2} \int_{\mathbb{R}^N} |\varphi|^{\alpha-1} \left(\frac{\partial\bar{\varphi}}{\partial t} \varphi + \frac{\partial\varphi}{\partial t} \bar{\varphi} \right) dx$$

$$= \frac{\alpha+2}{2} \left(\langle \frac{d\varphi}{\partial t}, |\varphi|^{\alpha-1}\varphi \rangle + \langle |\varphi|^{\alpha-1}\varphi, \frac{\partial\varphi}{\partial t} \rangle \right)$$

Even $(*)$ is formal since $|\varphi|^{\alpha-1}\varphi \notin H^1$

$$\frac{d}{dt}\mathcal{H}(\varphi(t)) = \frac{1}{2}(*) - \frac{1}{\alpha+2}(**)$$

$$= \frac{1}{2}\left(\left\langle \frac{d\varphi}{\partial t}, -\Delta\varphi - |\varphi|^\alpha \varphi \right\rangle + \left\langle -\Delta\varphi - |\varphi|^\alpha\varphi, \frac{d\varphi}{\partial t} \right\rangle \right)$$

$$= \frac{1}{2}\left(i \left\| \frac{d\varphi}{\partial t} \right\|_{L^2}^2 - i \left\| \frac{d\varphi}{dt} \right\|_{L^2}^2 \right) = 0$$

$\Rightarrow$ $\mathcal{H}(t)$ constant provided $(*)$ and $(**)$ become rigorous:

Consider $I_\varepsilon = (Id - \varepsilon\Delta)^{-2} : H^s \to H^{s-2}$
$\forall s \in \mathbb{R}$.

and $\varphi^\varepsilon(t)$ the (unique) solution of:

$$\begin{cases} i\dfrac{d\varphi^\varepsilon}{\partial t} = -\Delta\varphi^\varepsilon - I_\varepsilon\left[|I_\varepsilon \varphi_\varepsilon|^\alpha I_\varepsilon \varphi_\varepsilon \right] \\ \varphi^\varepsilon(0) \in H^2(\mathbb{R}^n) \quad \varphi^\varepsilon(0) \xrightarrow{H^1} \varphi(0) \end{cases}$$

Then $\varphi_\varepsilon \in C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2)$. and
$\varphi_\varepsilon(t) \xrightarrow{H^1} \varphi(t)$ as $\varepsilon \to 0$ $\forall t \in \mathbb{R}$

$(*)$ and $(**)$ are rigorous for φ_ε

So $\mathcal{H}(\varphi_\varepsilon(t)) \equiv \text{const}$

$\Rightarrow \lim\limits_{\varepsilon \to 0} \mathcal{H}(\varphi_\varepsilon(t)) \equiv \text{const}$

$\Rightarrow \quad \mathcal{H}(\varphi(t)) \equiv \text{const}$. <u>QED</u>.

<u>Proof of Theorem</u>: Fix t_0, assume $\exists\, \varepsilon > 0$, $\varphi_n \xrightarrow[H^1]{} \varphi_\lambda(t_0)$ and $t_n \geq t_0$ such that the solutions $\varphi_n(t)$ of (2) and $\varphi_n(t_0) = \varphi_n$ satisfy

(***) $$\inf_{\theta \in \mathbb{R}} \inf_{y \in \mathbb{R}^N} \| \varphi_n(t_n) - e^{i\theta} \varphi_\lambda(t_n, \cdot - y) \| \geq \varepsilon$$

But $$\mathcal{H}(\varphi_n(t_n)) = \mathcal{H}(\varphi_n(t_0)) \longrightarrow \mathcal{H}(\varphi_\lambda(t_0)) = \min_{\substack{\|U\|_{L^2}^2 = \mu \\ U \in H^1}} \mathcal{H}(U)$$

and $\|\varphi_n(t_n)\|_{L^2}^2 \longrightarrow \|\varphi_\lambda(t_0)\|_{L^2}^2 = \mu$.

So $v_n = \dfrac{\sqrt{\mu}}{\|\varphi_n(t_n)\|_{L^2}} \varphi_n(t_n)$ is a minimizing sequence for $\mathcal{H}(U)$ subject to $\|U\|_{L^2}^2 = \mu$. By concentration

Compactness $\exists\ v_{n_k}$ and y_k such that

$$v_{n_k}(\cdot - y_{n_k}) \xrightarrow{L^p} U_0 \text{ a minimizer}$$

and $\quad v_{n_k} \longrightarrow U_0 \quad 2 \leq p < \frac{2N}{N-2}$.

Since the minimizer is <u>unique</u> up to translation and rotations $\Rightarrow \exists\ \theta, y$ such that

$$U_0 = e^{i\theta} U_\mu(\cdot - y).$$

So $\quad v_{n_k} \xrightarrow{L^2} e^{i\theta} U_\mu(\cdot - y)$

and since $\left.\begin{array}{l} \mathcal{H}(v_{n_k}) \longrightarrow I_\mu \\ \|v_{n_k}\|_{L^{\alpha+2}}^{\alpha+2} \longrightarrow \|U_0\|_{L^{\alpha+2}}^{\alpha+2} \end{array}\right\} \Rightarrow$

$$\left.\begin{array}{l} \Rightarrow \|\nabla v_{n_k}\|_{L^2} \longrightarrow \|\nabla U_0\|_{L^2} \\ \text{and since } v_{n_k} \underset{L^2}{\rightharpoonup} \nabla U_0 \end{array}\right\} \Rightarrow \nabla v_n \xrightarrow{L^2} \nabla U_0$$

All in all $\quad v_{n_k} \xrightarrow{H^1} U_0 = e^{i\theta} U_\mu(\cdot - y)$

in contradiction with $(***)$. $\qquad$ <u>QED.</u>

Essential in the above proof of stability was the fact that minimizing sequences are precompact. This can be avoided sometimes like in the following general argument:

Another look at the Lyapunov method

Consider

$$(4) \quad \frac{dx}{dt} = F(t, x(t))$$

$x \in C(\mathbb{R}, X_1) \cap C^1(\mathbb{R}, X_2)$
$F : \mathbb{R} \times X_1 \to X_2$, X_1, X_2 real Banach spaces

Example Problem (2) can be cast in this form with $X_1 = H^1(\mathbb{R}^N)$ and $X_2 = H^{-1}(\mathbb{R}^N)$

Assume there exist a Lyapunov functional:

$$\mathcal{E} : X_1 \to \mathbb{R}$$

such that

(i) $\mathcal{E}$ is continuous

(ii) $x_0 \in X_1$ is a local minimum for $\mathcal{E}$ and $\exists\, \delta > 0,\ C > 0$ such that

$$\mathcal{E}(x) - \mathcal{E}(x_0) \geq C \|x - x_0\|_{X_1}^2 \quad \text{for } \|x - x_0\|_{X_1} \leq \delta$$

(iii) $\mathcal{E}(x(t))$ is nonincreasing along any solution of (4)

Then, provided eq (4) has solutions on $[0, \infty)$ whenever $\|x(0) - x_0\| < \delta_1$ for some $\delta_1 > 0$, $x(t) \equiv x_0$ is a stable equilibrium for (4).

<u>Sketch of proof</u>: $x(t) \equiv x_0$ is a solution of (4).
Indeed consider (4) with initial data $x(0) = x_0$ and $x : [0, \infty) \to X_1$ a solution of it.
Assume $x(t_1) \neq x_0$ for some $t_1 > 0$
Choose

$$t_2 = \inf \left\{ t > 0 : \|x(t) - x_0\|_{X_1} \geq \min \left[\frac{\delta}{2},\ \|x(t_1) - x_0\|_{X_1} \right] \right\}.$$

then $x(t_2) \neq x_0$ and.

$$\mathcal{E}(x(t_2)) - \mathcal{E}(x_0) \geq C \|x(t_2) - x_0\|_{X_1}^2 > 0$$

contradicts (iii).

Fix $\varepsilon > 0$, then by continuity of $\mathcal{E}$ at x_0 $\exists\ \delta(\varepsilon) < \delta$

$$|\mathcal{E}(x) - \mathcal{E}(x_0)| < C\varepsilon \quad \text{for } \|x - x_0\|_{X_1} < \delta(\varepsilon)$$

where $C > 0$ is given by (ii).

Now consider a sln $x(t)$ of (4) such that

$$\|x(0) - x_0\|_{X_1} < \delta(\varepsilon)$$

We have.

$$\mathcal{E}(x(0)) - \mathcal{E}(x_0) \overset{(iii)}{\geq} \mathcal{E}(x(t)) - \mathcal{E}(x_0) \overset{\text{by (ii)}}{\geq}$$
$$\geq C\|x(t) - x_0\|_{X_1}$$

Hence $\|x(t) - x_0\|_{X_1} \leq \dfrac{\mathcal{E}(x(0)) - \mathcal{E}(x_0)}{C} < \varepsilon$

So x_0 is stable.

There is one glitch! (ii) cannot be applied if $\|x(t) - x_0\| > \delta$. This however can be fixed:

- if $\|x(t_1) - x_0\| > \delta$ for some $t_1 > 0$ choose
$$t_2 = \inf\{t > 0 : \|x(t) - x_0\|_{X_1} > \delta\}$$
- then for $\varepsilon < \delta$ we get by above argument
$$\|x(t) - x_0\|_{X_1} < \varepsilon < \delta \quad \forall\, t \in [0, t_2] \text{ contradiction}$$

with the choice of t_2! QED

Remarks

1° (ii) is usually verified via

$$F(t,x) = F(x) = D\mathcal{E}\big|_x$$ in which case $X_2 = X_1^*$, and

$$DF\big|_{x_0} = D^2\mathcal{E}\big|_{x_0}$$ positive definite.

2° (ii) apparently excludes situations in which the local minima are not separated: $\exists\, x_n \to x_0$, $x_n \neq x_0$ and x_n local minimum for $\mathcal{E}$. However if the other local minima are due solely to the invariance of $\mathcal{E}$ under a group of symmetries $x \to Sx$ one can "mode out" the equivalence classes induced by the symmetry group and still be able to apply the above technique.

An example is the case of problem (2) see:

M. I. Weinstein: Lyapunov stability of grounds states of nonlinear dispersive evolution equations, Commun Pure Appl Math, Vol 39 (1986) p 51-68