Lecture 11 Math 555

Note Title 3/26/2008

3. Semigroup of Operators and Nonlinear Eq.

3.1 Continuous semigroups of operators. Infinitesimal generators.

3.2. Hille-Yosida; Lummer-Phillips, Stone Theorem.

3.3. Perturbations of infinitesimal generators

3.4. The Abstract Cauchy Problem

3.4. An application to the linear Schrödinger equation

3.5 An application to the nonlinear Schrödinger equation.

3.1. Continuous semigroups of operators: Infinitesimal generators.

<u>Def</u> (Semigroup of operators): X Banach space. $T(t): X \to X$ linear and <u>bounded</u> $0 \le t < \infty$ is called a semigroup if.

(i) $T(0) = I_d$

(ii) $T(t+s) = T(t)T(s) \quad \forall\, t, s \geq 0$

Def A semigroup of bounded linear operators $T(t)$ is called:

(a) uniformly continuous if

$$\lim_{t \searrow 0} \| T(t) - I_d \|_{\mathcal{B}(X,X)} = 0$$

(b) (strongly) continuous if

$$\lim_{t \searrow 0} \| T(t) x - x \| = 0 \quad \forall x \in X$$

Remark By (ii) we have:

(a) For uniformly continuous semigroup $T(t)$

$$\lim_{t \to s,\, t > s} \| T(t) - T(s) \|_{\mathcal{B}(X,X)} = 0$$

(b) For continuous semigroups

$$\lim_{t \to s,\, t > s} \| T(t) x - T(s) x \| = 0$$

Def The linear operator $A : D(A) \subseteq X \to X$ defined by:

$$D(A) = \left\{ x \in X : \lim_{t \searrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

$$Ax = \lim_{t \searrow 0} \frac{T(t)x - x}{t} \overset{\text{notation}}{=} \left. \frac{d^+ T(t)x}{dt} \right|_{t=0}$$

is called the infinitesimal generator of the semigroup $T(t)$.

Theorem 1 1° If $T(t)$ is a uniformly continuous semigroup then its infinitesimal generator is bounded.

2° Any linear bounded operator $A : X \to X$ generates a uniformly continuous semigroup of operators

Proof 1° Since $\lim_{\rho \searrow 0} \rho^{-1} \int_0^\rho T(s)\,ds = I_d$

we can choose ρ small enough so that

$$\| I_d - \rho^{-1} \int_0^\rho T(s)\,ds \|_{B(X,X)} < 1$$

$\Rightarrow \quad I_d - \left(I_d - \rho^{-1} \int_0^\rho T(s)\,ds \right)$ is invertible

$\Rightarrow \quad \rho^{-1} \int_0^\rho T(s)\,ds$ is invertible for $\rho > 0$ small.

Fix such a $\rho \Rightarrow \int_0^\rho T(s)\,ds$ is invertible.

$$
\begin{aligned}
(T(h) - I_d) \int_0^\rho T(s)\,ds &= \int_0^\rho T(h+s)\,ds - \int_0^\rho T(s)\,ds \\
&= \int_h^{\rho+h} T(s)\,ds - \int_0^\rho T(s)\,ds \\
&\overset{h \le \rho}{=} \int_\rho^{\rho+h} T(s)\,ds - \int_0^h T(s)\,ds.
\end{aligned}
$$

$$
\Rightarrow \quad h^{-1}(T(h) - I_d) = \left(h^{-1} \int_\rho^{\rho+h} T(s)\,ds - h^{-1} \int_0^h T(s)\,ds \right) \times \left(\int_0^\rho T(s)\,ds \right)^{-1}
$$

$$
\Rightarrow \quad \lim_{h \searrow 0} \frac{T(h) - I_d}{h} = \underbrace{(T(\rho) - I_d)\left(\int_0^\rho T(s)\,ds \right)^{-1}}_{\text{bounded linear operator.}}
$$

2° Define $T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = I + tA + \frac{t^2 A^2}{2} \dots$

The RHS is absolutely convergent in $\mathcal{B}(X, X)$.

Then $T(t)$ is a semigroup of operators (check) and

$$T(t) - I_d = \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} = tA\left(\sum_{n=0}^{\infty} \frac{(tA)^n}{(n+1)!}\right)$$

$$\Rightarrow \|T(t) - I_d\|_{B(X,X)} \leq |t|\,\|A\|_{B(X,X)} \cdot e^{t\|A\|_{B(X,X)}}$$

$$\Rightarrow \|T(t) - I_d\|_{B(X,X)} \xrightarrow{t \to 0} 0$$

Moreover

$$\left\|\frac{T(t) - I_d}{t} - A\right\|_{B(X,X)} = \left\|\frac{tA^2}{2} + \frac{tA^3}{3!} + \dots\right\| \leq \|A\| e^{t\|A\|}$$

$\Rightarrow \lim_{t \to 0} \frac{T(t) - I_d}{t} = A \Rightarrow A$ is the infinitesimal generator of $T(t) = e^{tA}$. QED

<u>Corollary</u> If $T(t)$ is a uniformly continuous semigroup of operators then

a) $\exists\, \omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$

b) $\exists !\, A \in B(X,X)$ such that $T(t) = e^{tA}$

c) $t \to T(t)$ is differentiable in norm and
$$\frac{dT(t)}{dt} = AT(t) = T(t)A$$

Theorem 2. If $T(t)$ is a continuous semigroup of operators then $\exists\ \omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq M e^{\omega t} \quad \forall t \geq 0$$

Proof $\exists\ \delta > 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq M \quad \forall\ 0 \leq t \leq \delta$$

otherwise $\exists\ t_n \to 0$ such that

$$\|T(t_n)\| > n$$

But $\|T(t_n)x\|$ is bounded for every $x \in X$ because $\|T(t_n)x - x\| \xrightarrow{n\to\infty} 0$. By unif bdd principle $\Rightarrow$ $\|T(t_n)\|$ is bdd contradiction.

Choose $\omega = \dfrac{\ln M}{\delta} \geq 0$.

For $t \geq 0$ $\exists\ n \in \mathbb{N}, \tau \geq 0$: $t = n\delta + \tau$, $\tau < \delta$

$$\Rightarrow \|T(t)\| = \|T(\tau)T(n\delta)\| = \|T(\tau)\, T(\delta)^n\|$$
$$\leq M^{n+1} \leq M\, M^{t/\delta} = M e^{\omega t}$$

Theorem 3 If $T(t)$ is a continuous semigroup of operators with infinitesimal generator A then:

a) $$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x\, ds = T(t)x \quad \forall x \in X$$

b) $\int_0^t T(s)x\,ds \in D(A)$ $\forall x \in X, t > 0$ and

$$A\left[\int_0^t T(s)x\,ds\right] = T(t)x - x.$$

c) $\forall x \in D(A) \Rightarrow T(t)x \in D(A)$ and

$$\frac{d}{dt} T(t)x = A T(t)x = T(t)Ax$$

d) $\forall x \in D(A)$

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax\,d\tau = \int_s^t AT(\tau)x\,d\tau$$

<u>Proof</u>: a) follows from continuity of $t \mapsto T(t)x$

b) For $x \in X$, $h > 0$ we have

$$\frac{T(h) - \mathrm{Id}}{h} \int_0^t T(s)x\,ds = \frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^t T(s)x\,ds$$

$h \to 0$ proves b).

c) For $x \in D(A)$, $h > 0$ we have

$$\frac{T(h)-Id}{h} T(t)x = T(t) \frac{T(h)-Id}{h} x \xrightarrow{h\to 0} T(t)Ax$$

$\Rightarrow$ $T(t)x \in D(A)$ and

$$\lim_{h\searrow 0} \frac{T(t+h)x - T(t)x}{h} = AT(t)x = T(t)Ax$$

But for $t>0$:

$$\lim_{h\searrow 0}\left[\frac{T(t)x - T(t-h)x}{h} - T(t)Ax\right] =$$

$$= \lim_{h\searrow 0} T(t-h)\left[\frac{T(h)x - x}{h} - Ax\right] +$$

$$\lim_{h\searrow 0}\left[T(t-h)Ax - T(t)Ax\right] = 0$$

So for $t>0$:

$$\lim_{h\to 0} \frac{T(t+h)x - T(t)x}{h} = T(t)Ax = AT(t)x$$

c) is finished and by integrating it we get d) <u>QED</u>

Corollary If A is the infinitesimal generator of the continuous semigroup of operators $T(t)$ then

$$\overline{D(A)} = X \text{ and } A \text{ is closed}$$

Proof For $x \in X$ set $x_n = n \int_0^{1/n} T(s) x \, ds$

by b) $x_n \in D(A)$, by a) $x_n \to x \Rightarrow \overline{D(A)} = X$

Let $x_n \in D(A)$ $x_n \to x \in X$ and $A x_n \to y \in X$. Then

$$\underbrace{T(t) x_n - x_n}_{\text{converges to } T(t)x - x} = \int_0^t \underbrace{T(s) A x_n}_{\text{converges unif on bdd intervals to } T(s)y} ds.$$

$\Rightarrow$ $T(t)x - x = \int_0^t T(s) \, y \, ds.$

$\Rightarrow$ $\lim_{t \downarrow 0} \frac{T(t)x - x}{t}$ exists and $= y$

$\Rightarrow$ $x \in D(A)$ and $Ax = y \Rightarrow A$ closed QED

Remark If $T(t)$ is a continuous semigroup with infinitesimal generator A and $D(A^n)$ is the domain of A^n, then

$$\overline{\bigcap_{n=1}^{\infty} D(A^n)} = X.$$

Proof $Y = \{ \int_0^\infty \varphi(s) T(s) x \, ds \mid x \in X, \ \varphi \in C_0^\infty(\mathbb{R}_+) \}$
Then $Y \subseteq \bigcap_{n=1}^{\infty} D(A^n)$ and $\overline{Y} = X$.

Theorem 4 If $T(t)$ and $S(t)$ are continuous semigroups of operators with the same generator then $T(t) \equiv S(t)$.

Proof Let A be their common infinitesimal generator and $x \in D(A)$. Then for $t > 0$, $0 < s < t$

$$s \longmapsto T(t-s) S(s) x \text{ is differentiable}$$

and

$$\frac{d}{ds} T(t-s) S(s) x = -T(t-s) A S(s) x + T(t-s) A S(s) x = 0.$$

$\Rightarrow$ $s \longmapsto T(t-s) S(s) x$ is constant $\Rightarrow$ for $s = 0$ and $s = t$: $T(t) x = S(t) x$ QED.