3.2. Hille-Yosida, Lumer-Phillips and Stone's Theorems

Theorem (Hille-Yosida) A is the infinitesimal generator of a continuous semigroup $T(t)$, $t \geq 0$, of contractions, $\|T(t)\| \leq 1$ $\forall t \geq 0$, if and only if:

(i) A is closed and $\overline{D(A)} = X$

(ii) $(0, \infty) \subseteq \rho(A)$ = resolvent set of A and

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0$$

where $R(\lambda, A) = (\lambda I_d - A)^{-1}$

Proof "$\Longrightarrow$" (i) is a Corollary in Lect. 11.

For (ii) define for $\lambda > 0$

$$(1) \quad R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt$$

Since $\|T(t)x\| \leq \|x\| \; \forall t \geq 0$ the integral is absolutely convergent and $R(\lambda) : X \to X$ is a linear, bounded operator with:

$$\|R(\lambda)x\| \leq \frac{1}{\lambda}\|x\|$$

For $h > 0$

$$\frac{T(h) - Id}{h} R(\lambda)x =$$

$$= \frac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda t} T(t)x \, dx$$

$$- \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} T(t)x \, dx$$

RHS $\longrightarrow \lambda R(\lambda)x - x$ as $h \searrow 0$

$\Rightarrow R(\lambda)x \in D(A) \quad \forall \lambda > 0, x \in X$

and

$$A R(\lambda)x = (\lambda R(\lambda) - Id)x$$

$\Rightarrow (\lambda Id - A) R(\lambda) = Id.$

But from (1) A and $R(\lambda)$ commute on $D(A)$

$\Rightarrow \quad R(\lambda)(\lambda I_d - A)x = x \quad$ on $D(A)$

So $\quad R(\lambda) = (\lambda I_d - A)^{-1}$

"$\Leftarrow$" The Yosida approx of A: $\lambda > 0$

$$A_\lambda = \lambda A R(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I_d$$

Properties of Yosida approximation:

$$1^\circ \quad \lim_{\lambda \to \infty} A_\lambda x = Ax \qquad \forall x \in D(A)$$

2° A_λ is a bounded linear op on X and

$$\| e^{tA_\lambda} x - e^{tA_\mu} x \| \leq t \| A_\lambda x - A_\mu x \|$$

$\forall x \in X$, $t > 0$, $\lambda, \mu > 0$.

For 1° $\| \lambda R(\lambda, A) x - x \| = \| A R(\lambda, A) x \|$

$$= \| R(\lambda, A) A x \| \leq \frac{1}{\lambda} \| Ax \| \xrightarrow{\lambda \to \infty} 0$$

for $x \in D(A)$

$\Rightarrow \quad \lambda R(\lambda, A) x \xrightarrow{x \to \infty} x$, $x \in X$ by density of $D(A)$ in X.

Now $\lambda A R(\lambda, A) x = A R(\lambda, x) A x \xrightarrow{\lambda \to \infty} Ax$.

Part 2°: $\|e^{tA_\lambda}\| = e^{-t\lambda}\|e^{t\lambda^2 R(\lambda,A)}\|$

$\leq e^{-t\lambda} e^{t\lambda^2 \|R(\lambda,A)\|}$

$\leq 1.$

$\|e^{tA_\lambda x} - e^{tA_\nu x}\| =$

$\|\int_0^1 \frac{d}{ds}(e^{tsA_\lambda} e^{t(1-s)A_\nu} x)\,ds\|$

$\leq \int_0^1 t\|e^{tsA_\lambda} e^{t(1-s)A_\nu}(A_\lambda x - A_\nu x)\|ds$

Note that $\leq t\|A_\lambda x - A_\nu x\|$

A_λ, A_ν commute and so do $e^{A_\lambda t}, e^{A_\nu t}$ $t \geq 0$.

Back to the Theorem:

$\|e^{tA_\lambda}x - e^{tA_\nu}x\| \leq t\|A_\lambda x - A_\nu x\|$

$\leq t\|A_\lambda x - Ax\| + t\|Ax - A_\nu x\|$

$\Rightarrow$ for $x \in D(A)$ $e^{tA_\lambda}x$ converges uniformly in t on bdd intervals. Denote

$e^{tA_\lambda}x = T(t)x$

Since $\|e^{tA_n}\| \le 1$ $t \ge 0 \Rightarrow \|T(t)x\| \le \|x\|$ $\forall x \in D(A)$ and $t \ge 0$. So $T(t)$ can be extended to a linear, bounded operator on $\overline{D(A)} = X$ and the continuity in t follows from the continuity of e^{tA_n} together with the unif convergence in t on bdd intervals.

For $\lim_{t \to 0} \frac{T(t)x - x}{t} = Ax$ $\forall x \in D(A)$ calculate

$$T(t)x - x = \lim_{n \to \infty} \left(e^{tA_n} x - x \right)$$

$$= \lim_{n \to \infty} \int_0^t e^{sA_n} A_n x \, ds$$

$$= \int_0^t T(s) A x \, ds.$$

If B = infinitesimal generator of $T(t)$ then for $x \in D(A)$ we have $x \in D(B)$ and

$Bx = Ax$. (divide by t and do $t \downarrow 0$ in the above)

By the first part of the theorem $1 \in \rho(B) \cap \rho(A)$

So $(I - B)D(A) = (I - A)D(A) = X$

$\Rightarrow D(A) = (I-B)^{-1} X = D(B)$.

So $A = B$ QED

Def (Dual set) For $x \in X$ define

$$F(x) = \{ x^* \in X^* \mid \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2 \}$$

Def (Dissipative operator) $A : D(A) \to X$ linear
$\subseteq X$
is called dissipative if $\forall x \in D(A)\ \exists x^* \in F(x)$:

$$\operatorname{Re} \langle Ax, x^* \rangle \leq 0$$

Example $\Delta : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$

Theorem Linear operator A is dissipative if and only if $\|(\lambda I - A)x\| \geq \lambda \|x\| \quad \forall x \in D(A)$,
$\lambda > 0$.

(See Pazy, Semigroups of linear operators, p. 14)

<u>Theorem</u> (Lumer-Phillips) Let $A : D(A) \to X$ linear with $\overline{D(A)} = X$

(a) If A is dissipative and $\exists \lambda_0 > 0$ such that Range $(\lambda_0 I_d - A) = X$ then A is the infinitesimal generator of a continuous semigroup of contractions: $T(t)$, $\|T(t)\| \leq 1$, $t \geq 0$.

(b) If A is the infinitesimal generator of a continuous semigroup of contractions: $T(t)$, $\|T(t)\| \leq 1$ $\forall t \geq 0$, then A is dissipative and Range $(\lambda I_d - A) = X$ $\forall \lambda > 0$. Moreover $Re \langle Ax, x^* \rangle \leq 0$ $\forall x \in D(A)$, $x^* \in F(x)$.

<u>Proof</u> By previous th:

$$\|(\lambda_0 I_d - A)x\| \geq \lambda_0 \|x\| \quad (*)$$

$\Rightarrow$ $\lambda_0 I_d - A : D(A) \to X$ is one to one and onto $\Rightarrow$ $R(\lambda_0; A) = (\lambda_0 I_d - A)^{-1} : X \to D(A)$ exists and it is continuous because by $(*)$ with $x = R(\lambda_0; A)y$

$$\|R(\lambda_0; A)y\| \leq \frac{1}{\lambda_0} \|y\| \quad \forall y \in X.$$

$\Rightarrow$ $\lambda_0 \in \rho(A)$ and $(\lambda_0 I_d - A)^{-1}$ continuous $\Rightarrow$ $(\lambda_0 I_d - A)^{-1}$ closed $\Rightarrow$ $\lambda_0 I_d - A$ closed $\Rightarrow$ A closed

Part (i) of Hille-Yosida's Theorem has been verified.

For part (ii) we do a connectivity argument:

Let $\Lambda = \{ \lambda \in (0,\infty) \mid \lambda \in \rho(A) \} \subseteq (0,\infty)$

$\lambda_0 \in \Lambda \Rightarrow \Lambda \neq \emptyset$.

Λ open because $\rho(A)$ is open

Λ closed in $(0,\infty)$: Let $\{\lambda_n\} \subseteq \Lambda$

$$\lambda_n \to \lambda > 0$$

Since $\lambda_n \in \rho(A) \Rightarrow \forall y \in X \; \exists x_n \in D(A)$:

$$(\lambda_n Id - A) x_n = y. \quad (**)$$

From (*)

$$\|x_n\| \leq \frac{1}{\lambda_n} \|y\| \leq C \text{ for some } C > 0$$

and

$$\lambda_m \|x_n - x_m\| \leq \| \lambda_m (x_n - x_m) - A(x_n - x_m) \|$$
$$= \| (\lambda_m - \lambda_n) x_n \| \leq |\lambda_m - \lambda_n| \cdot C$$

$\Rightarrow \{x_n\}$ Cauchy $\Rightarrow x_n \to x$ in $X \overset{**}{\Rightarrow}$

$$A x_n \to y + \lambda x.$$

A closed $\Rightarrow x \in D(A)$ and $Ax = y + \lambda x$

$\Rightarrow y \in \text{Range}(\lambda Id - A)$

$y \in X$ arbitrary $\Rightarrow \text{Range}(\lambda Id - A) = X \xrightarrow[\lambda_0]{\text{as for}} \lambda \in \rho(A)$

$\Rightarrow \lambda \in \Lambda$. So Λ is open and closed in $(0,\infty)$ connected $\Big\} \Rightarrow \Lambda = (0,\infty)$ and $\Lambda \neq \emptyset$

Hence (ii) from Hille-Yosida Th is checked (The $\|R(\lambda; A)\| \leq \frac{1}{\lambda}$, $\lambda > 0$ follows from previous Th).

Both (i), (ii) are satisfied $\overset{\text{Hille-Yosida}}{\Rightarrow}$ (a).

(b) Range $(\lambda I_d - A) = X$, $\lambda > 0$ follows from $\lambda \in \rho(A)$ $\forall \lambda > 0$ see (ii) in Hille-Yosida.
Let $x \in D(A)$ and $x^* \in F(x)$ then:

$$|\langle T(t)x, x^* \rangle| \leq \|T(t)x\| \, \|x^*\| \leq \|x\| \, \|x^*\| = \|x\|^2$$

$$\Rightarrow \text{Re} \langle T(t)x - x, x^* \rangle = \text{Re} \langle T(t)x, x^* \rangle - \|x\|^2 \leq 0$$

Divide by $t > 0$ and let $x \searrow 0$

$$\Rightarrow \text{Re} \langle Ax, x^* \rangle \leq 0$$

Since $x \in D(A)$ and $x^* \in F(x)$ arbitrary (b) follows ///

<u>Application</u>: $\Delta : H^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ generates a continuous semigroup of contractions:

L^2 Hilbert $\Rightarrow$ $F(\varphi) = \{\varphi\}$.

$Re \langle \Delta\varphi, \varphi \rangle = Re - \langle \nabla\varphi, \nabla\varphi \rangle \leq 0$
$\Rightarrow$ Δ dissipative.

Consider the eq $\varphi - \Delta\varphi = \psi$ for $\psi \in L^2$
Fourier Transform:

$$\hat{\varphi}(\xi) + |\xi|^2 \hat{\varphi}(\xi) = \hat{\psi}(\xi)$$

$$\Rightarrow \quad \hat{\varphi}(\xi) = \frac{\hat{\psi}(\xi)}{1+|\xi|^2} \quad \text{and} \quad \varphi \in H^2(\mathbb{R}^n)$$

because $(1+|\xi|^2)\hat{\varphi}(\xi) \in L^2(\mathbb{R}^n) \Rightarrow \begin{cases} \hat{\varphi}(\xi) \in L^2(\mathbb{R}^n) \\ \xi_j \hat{\varphi}(\xi) \in L^2(\mathbb{R}^n) \Rightarrow \\ \xi_i \xi_j \hat{\varphi}(\xi) \in L^2(\mathbb{R}^n) \end{cases}$

Inverse $\Rightarrow$ F.T. $\begin{cases} \varphi \in L^2(\mathbb{R}^n) \\ \dfrac{\partial \varphi}{\partial x_j} \in L^2(\mathbb{R}^n) \\ \dfrac{\partial^2 \varphi}{\partial x_j \partial x_i} \in L^2(\mathbb{R}^n \end{cases}$

So for any $\psi \in L^2(\mathbb{R}^n)$, $(I_d - \Delta)\varphi = \psi$ has a solution in $H^2(\mathbb{R}^n)$: $\varphi(x) = \left(\frac{\hat{\psi}(\xi)}{1+|\xi|^2} \right)^{\vee}(x)$

$\Rightarrow$ Range $(I_d - \Delta) = L^2(\mathbb{R}^n)$

Hence (a) is verified $\Rightarrow$ Δ generates a continuous semigroup of contractions on L^2.

<u>Corollary</u> Let $A : D(A) \to X$ linear, closed and $\overline{D(A)} = X$. If both A and A^* are dissipative, then A is the infinitesimal generator of a continuous semigroup of contractions.

<u>Proof</u> It suffices to show Range $(I_d - A) = X$

Assume not. Then

$$\underset{A \text{ closed}}{\|(I_d - A)x\| \geq \|x\|} \Rightarrow \underset{\text{closed} \neq X}{\text{Range}(I_d - A)}$$

$\overset{\text{Hahn}}{\underset{\text{Banach}}{\Rightarrow}}$ $\exists\, x^* \in X^*$ $x^* \neq 0$ so that

$$\langle (I_d - A)x, x^* \rangle = 0 \quad \forall\, x \in D(A).$$

$\Rightarrow$ $x^* \in D((I_d - A)^*)$ and $(I_d^* - A^*)x^* = 0$

but A^* dissipative $\Rightarrow 0 = \|(I_d^* - A^*)x^*\| \geq \|x^*\| \Rightarrow \|x^*\| = 0$ $\rightarrow\leftarrow$

Remark If A is dissipative and $\overline{D(A)} = X$ then A is closable.

Theorem (Characterization of infinitesimal generators of semigroup of operators)
A linear operator $A : D(A) \to X$ is the infinitesimal generator of a continuous semigroup of operators $T(t)$, $t \geq 0$, satisfying $\|T(t)\| \leq M e^{wt}$ if and only if:

(i) A is closed and $\overline{D(A)} = X$

(ii) $(w, \infty) \subseteq \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - w)^n} \quad \text{for } \lambda > 0, \; n = 1, 2, \dots$$

Proof see Pazy, page 20.

Def $T(t) : X \to X$ linear and bounded, $t \in \mathbb{R}$ is called a group of bounded operators if:

(i) $T(0) = I_d$

(ii) $T(t+s) = T(t)T(s) \quad \forall\, t, s \in \mathbb{R}$

If in addition it satisfies:

(iii) $\lim_{t \to 0} T(t)x = x$

Then $T(t)$ is called a continuous group of bounded operators.

<u>Def</u> If $T(t), t \in \mathbb{R}$ is a continuous group of bounded operators then the linear operator

$$A : D(A) \to X$$

$$D(A) = \{ x \in X \mid \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \}$$

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}$$

is called the infinitesimal generator of $T(t)$.

<u>Remark</u> (1) If $T(t), t \in \mathbb{R}$ is a continuous group of bounded operators with infinitesimal generator A then

$$T_+(t) = T(t) \quad t \geq 0$$

$$T_-(t) = T(-t) \quad t \geq 0$$

are continuous semigroups of bounded operators with infinitesimal generators A respectively $-A$.

(2) The converse: If A and $-A$ are infinitesimal generators of continuous semigroups of operators denoted by $T_+(t), T_-(t), t \geq 0$ then

$$T(t) = \begin{cases} T_+(t) & t \geq 0 \\ T_-(-t) & t < 0 \end{cases}$$

is a continuous group of bounded operators, will be proven in the next theorem:

Theorem A is the infinitesimal generator of a continuous group of bounded operators $T(t)$ satisfying $\|T(t)\| \le M e^{\omega|t|}$, $t \in \mathbb{R}$, iff:

(i) A is closed and $\overline{D(A)} = X$

(ii) $(-\infty, -\omega) \cup (\omega, \infty) \subseteq \rho(A)$ and

$$\|R(\lambda, A)^n\| \le M(|\lambda| - \omega)^{-n}, \quad n = 1, 2, \ldots$$
$$\lambda \in \mathbb{R}, \ |\lambda| > \omega$$

Proof Note that

$$R(-\lambda, -A) = -R(\lambda, A) \qquad (2)$$

Now "$\Rightarrow$" follows from Remark part (1) and the characterization of infinitesimal generators of continuous semigroups

"$\Leftarrow$" (i), (ii), (2) + characterization of infinitesimal generators of continuous semigroups of op $\Rightarrow$ A, $-A$ generate such semigroups. Denote them by $T_+(t)$, $T_-(t)$, $t \ge 0$

Note $T_+(t) = \lim_{\lambda \to \infty} e^{tA_\lambda}$, $T_-(t) = \lim_{\nu \to \infty} e^{-tA_\nu}$

where $A_\lambda = \lambda A R(\lambda, A)$ is the Yosida approx.

Then T_+ and T_- commute because e^{tA_n} and e^{-tA_n} commute.

Denote $W(t) = T_+(t)T_-(t)$ then for $x \in D(A) = D(-A)$

$$\frac{W(t)x - x}{t} = T_-(t)\frac{T_+(t)x - x}{t} + \frac{T_-(t)x - x}{t}$$

$$\longrightarrow I_d Ax + (-A)x = 0 \text{ as } t \searrow 0$$

$\Rightarrow W(t)$ is generated by $0 \Rightarrow W(t) = I_d$.

So $T_-(t) = (T_+(t))^{-1}$.

Now let:

$$T(t) = \begin{cases} T_+(t) & t \geq 0 \\ T_-(-t) = (T_+(-t))^{-1} & \text{for } t < 0 \end{cases}$$

Then (i) and (ii) from the definition of a group are obvious and for all $x \in X$

$$\lim_{t \searrow 0} T(t)x = x = \lim_{h \searrow 0} T_-(h)x = \lim_{t \nearrow 0} T_-(-t)x = \lim_{t \nearrow 0} T(t)x$$

So $\lim_{t \to 0} T(t)x = x$ and (iii) is checked.

Analogously, for $x \in D(A) = D(-A)$ we have

$$\lim_{t \searrow 0} \frac{T(t)x - x}{t} = Ax = \lim_{t \nearrow 0} \frac{T(t)x - x}{t}$$

$\Rightarrow$ the infinitesimal generator of $T(t)$ might be an extension of A, called B, but then for $x \in D(B)$

$\lim_{t \searrow 0} \frac{T(t)x - x}{t}$ exists and equals $Bx \Rightarrow \lim_{t \searrow 0} \frac{T_+(t)x - x}{t}$ exists $\Rightarrow x \in D(A) \Rightarrow A = B$. QED

Def (adjoint semigroup) If $T(t) : X \to X \quad t \geq 0$ is a semigroup of bounded operators then

$$T(t)^* : X^* \to X^*$$

is a semigroup of bounded operators called the adjoint semigroup.

Remark (1) If $T(t) : X \to X$, $t \geq 0$ is a continuous semigroup of bounded operator then $T(t)^*$ is not necessarilly continuous. However

$$T(t)^* \Big|_{Y^*} : Y^* \to Y^* \subseteq X^*$$

where $Y^* = \overline{D(A^*)}$, A^* = the adjoint of the infinitesimal generator of $T(t)$, is a continuous semigroup of operators with infinitesimal generator

$$A^* \Big|_S : S \to Y^*$$

where $S = \{ x^* \in D(A^*) \mid A^* x^* \in Y^* \}$

(2) If X is reflexive then from $\overline{D(A)} = X$ we get $\overline{D(A^*)} = X^*$.

Theorem (Stone) A is the infinitesimal generator of a continuous group of unitary operators on a Hilbert space H if and only if iA is self adjoint.

Proof "$\Rightarrow$" Let $U(t)$ be the group generated by A ($U(t)$ unitory $\overset{def}{\Leftrightarrow}$ $U^*(t) = U(t)^{-1}$). Then for $x \in D(A) = D(-A)$

$$-Ax = \lim_{t \searrow 0} \frac{U(-t)x - x}{t} = \lim_{t \searrow 0} \frac{U^*(t)x - x}{x} = A^* x$$

$\Rightarrow -A = A^* \Rightarrow (iA)^* = (iA) \Rightarrow iA$ is self-adjoint

"$\Leftarrow$" iA self adjoint $\Rightarrow \overline{D(A)} = H$ and $-A = A^*$.

Then

$$\langle Ax, x \rangle = \langle x, A^* x \rangle = -\overline{\langle Ax, x \rangle}$$

$\Rightarrow Re \langle Ax, x \rangle = 0 \Rightarrow A$ is dissipative

$Re \langle A^* x, x \rangle = Re - \langle Ax, x \rangle = 0 \Rightarrow A^*$ is dissipative

A, A^* dissipative $\Rightarrow A$ generates a continuous semigroup of contractions

$A^*, A^{**} = A$ dissipative $\Rightarrow A^* = -A$ generates a conti. semigroup of contraction

$A, -A$ generate continuous semigroups $\Rightarrow A$ generate a

Continuous group (of contractions). $U(t)$, $t \in \mathbb{R}$.

$$U(t) = \begin{cases} U_+(t) & t \geq 0 \\ U_-(-t) & t < 0 \end{cases},$$

where $U_+(t)$ is generated by A while $U_-(t), t \geq 0$ is generated by $-A = A^*$

Since for $t > 0$

$$(U(t))^{-1} = U(-t) = U_-(t) = U^*(t)$$

$\Rightarrow U(t)$ is unitary. QED.