Lecture 13 Math 555

Note Title 4/7/2008

Summary

— 3.3 Perturbations of infinitesimal generators

— 3.4 Abstract Cauchy Problem

3.3. Perturbations of infinitesimal generators

<u>Theorem</u> (perturbation by bounded operators)
X Banach space, A infinitesimal generator of C_0 semigroup $T(t)$, $\|T(t)\| \le M e^{\omega t}$ $t \ge 0$.
If $B: X \to X$ is linear and bounded then $A+B$ is the infinitesimal generator of a C_0 semigroup $S(t)$, $\|S(t)\| \le M e^{(\omega + M\|B\|)t}$, $t \ge 0$.

<u>Proof</u> Consider the integral equation:

$$(1) \quad S(t) = T(t) + \int_0^t T(t-s) B S(s)\, ds$$

We are going to show:

1° For any $\sigma > \omega + M\|B\|$ eq (1) has a unique solution in the Banach space

$$C^{(\sigma)} = \left\{ U \in C\big([0,\infty), B(X,X)\big) \;\Big|\; \sup_{t \geq 0} e^{-\sigma t} \|U(t)\|_{B(X,X)} < \infty \right\}.$$

and $\|S(t)\| \leq M e^{(\omega + M\|B\|)t}$

2° The solution in part 1° is a C_0 semigroup of operators on X with infinitesimal generator

$$A + B : D(A) \longrightarrow X.$$

Proof of 1° Let $K : C^{(\sigma)} \longrightarrow C^{(\sigma)}$

$$K\,U(t) = \int_0^t T(t-s)\, B\, U(s)\, ds.$$

where $C^{(\sigma)}$ is endowed with the norm:

$$\|U(t)\|_\sigma = \sup_{t \geq 0} e^{-\sigma t} \|U(t)\|_{B(X,X)}$$

Then K is a contraction:

$$e^{-\delta t} \| (KU_1)(t) - (KU_2)(t) \|_{B(X,X)} \leq$$

$$e^{-\delta t} \int_0^t M e^{\omega(t-s)} \|B\| e^{\delta s} \|U_1 - U_2\|_\delta \, ds$$

$$\leq \|U_1 - U_2\|_\delta \frac{M \|B\|}{\delta - \omega} \left(1 - e^{(\omega-\delta)t}\right)$$

$$\Rightarrow \| KU_1 - KU_2 \|_\delta \leq \frac{M\|B\|}{\delta - \omega} \|U_1 - U_2\|_\delta$$

$\Rightarrow K_T U(t) = T(t) + KU(t)$ is also a contraction and has a unique fixed point:

$S = K_T U = T + KS \iff$

$$S(t) = T(t) + \int_0^t T(t-s) B S(s) \, ds.$$

Apply the $B(X,X)$ norm to the above:

$$\|S(t)\| \leq \|T(t)\| + \int_0^t \|T(t-s)\| \|B\| \|S(s)\| \, ds$$

$$\leq M e^{\omega t} + \int_0^t M e^{\omega(t-s)} \|B\| \|S(s)\| \, ds$$

By Generalized Gronwall inequality

$\|S(t)\| \leq x(t)$ where $x(t)$ is the sln of

$$\begin{cases} x(0) = M \\ x'(t) = (\omega + M\|B\|)\,x(t) \end{cases}$$

$$\Leftrightarrow x(t) = M e^{(\omega + M\|B\|)t}$$

Proof of 2^0 Semigroup properties:

(i) $S(0) = T(0) = I_d$ ✓

(ii) $S(t)S(t_0) = S(t+t_0)$ $\forall\, t, t_0 \geq 0$?

Fix $t_0 \geq 0$, Multiply (1) to the right by $S(t_0)$.
Then $V(t) = S(t)S(t_0)$ $t \geq 0$ satisfies:

(2) $V(t) = T(t)S(t_0) + (K\,V)(t)$

But from (1) with $t \mapsto t+t_0$ and $W(t) = S(t+t_0), t \geq 0$
we have

(3) $$W(t) = T(t+t_0) + \int_0^{t+t_0} T(t+t_0-s)\, B\, S(s)\, ds$$

$$= T(t)T(t_0) + \int_0^{t_0} T(t)T(t_0-s)\, B\, S(s)\, ds$$

$$+ \int_{t_0}^{t+t_0} T(t+t_0-s)\, B\, S(s)\, ds$$

$$= T(t)S(t_0) + \int_0^t T(t-s')\, B\, \overbrace{S(s'+t_0)}^{W(s')}\, ds'$$

$$= T(t)S(t_0)x + (KW)(t)$$

So V and W satisfy the same equation involving the contraction K. By uniqueness (ii) is satisfied.

For any $x \in X$ we have

$$\lim_{t \searrow 0} \frac{1}{t} \int_0^t T(t-s)\, B\, S(s)x\, ds = Bx$$

Indeed for $t > 0$ such that $\|S(s)x - x\| < \varepsilon \quad \forall\, 0 \le s \le t$

$$\left\| \frac{1}{t} \int_0^t T(t-s)\, B(S(s)x - x)\, ds \right\| \le \frac{1}{t} \int_0^t Me^{\omega(t-s)} \|B\| \cdot \varepsilon\, ds$$

$$\le Me^{\omega t} \varepsilon$$

Hence

$$\lim_{t \searrow 0} \frac{1}{t} \int_0^t T(t-s) B S(s) x \, ds = \lim_{t \searrow 0} \frac{1}{t} \int_0^t T(t-s) B x \, ds$$

$$= Bx$$

Now

$$\frac{S(t)x - x}{t} = \frac{T(t)x - x}{t} + \frac{1}{t} \int_0^t T(t-s) B S(s) x \, ds$$

and $\lim_{t \searrow 0} \frac{S(t)x - x}{t}$ exists iff $\lim_{t \searrow 0} \frac{T(t)x - x}{x}$ exists and

$$\lim_{t \searrow 0} \frac{S(t)x - x}{t} = \overset{\nearrow Ax}{\lim_{t \searrow 0} \frac{T(t)x - x}{t}} + Bx$$

Hence $A + B : D(A) \to X$ is the infinitesimal generator of S.

The theorem is completely proven! QED

Corollary: If A is the infinitesimal generator of the C_0 semigroup of operators $T(t): X \to X$, $\|T(t)\| \le Me^{\omega t}$ $t \geq 0$, and $B: X \to X$ is linear and bounded then the C_0 semigroup generated by $A+B$, $S(t)$ $t \geq 0$ satisfies:

$$\|S(t) - T(t)\| \le Me^{\omega t}\left(e^{M\|B\|t} - 1\right)$$

Proof from (1) we have

$$\|S(t) - T(t)\| \le \int_0^t \|T(t-s)\| \, \|B\| \, \|S(s)\| \, ds$$

$$\le \int_0^t Me^{\omega(t-s)} \|B\| \, M e^{(\omega + M\|B\|)s} \, ds$$

$$= Me^{\omega t} \int_0^t M\|B\| e^{M\|B\|s} \, ds$$

$$= Me^{\omega t}\left(e^{M\|B\|t} - 1\right) \qquad \text{QED}$$

Theorem (perturbations of infinitesimal generators of contraction semigroups) Let A be the infinitesimal generator of a C_0 semigroup of contractions. Let B be dissipative satisfying $D(B) \supset D(A)$ and

$$\|Bx\| \leq \alpha \|Ax\| + \beta \|x\| \quad \text{for } x \in D(A)$$

where $0 \leq \alpha < 1$ and $\beta \geq 0$. Then $A+B$ is the infinitesimal generator of a C_0 semigroup of contractions.

Proof By Lumer-Phillips:

$$\overline{D(A)} = X$$

$$\forall x \in D(A), \forall x^* \in F(x) \quad \mathrm{Re}\langle Ax, x^* \rangle \leq 0$$

and $\mathrm{Range}(I - A) = X$.

Consequently: $\overline{D(A+B)} = \overline{D(A)} = X$

$$\forall x \in D(A+B) = D(A) \quad \exists x^* \in F(x) : \mathrm{Re}\langle Bx, x^* \rangle \leq 0$$

$\Rightarrow \mathrm{Re}\langle (A+B)x, x^* \rangle \leq 0 \Rightarrow A+B$ is dissipative

It remains to show that $\mathrm{Range}(I - A - B) = X$ after which Lumer-Phillips implies the conclusion

Lemma Let A, B be linear operators such that $D(B) \supseteq D(A)$ and $A + tB$ is dissipative for all $0 \le t \le 1$. If

$$\|Bx\| \le \alpha \|Ax\| + \beta \|x\| \text{ for } x \in D(A)$$

where $0 \le \alpha < 1$, $\beta \ge 0$, and for some $t_0 \in [0,1]$ $\text{Range}(I - (A + t_0 B)) = X$ then for all $t \in [0,1]$

$$\text{Range}(I - (A + tB)) = X$$

Proof of Lemma: It suffices to show that $\exists\, \delta > 0$ independent of t_0 such that

$$\text{Range}(I - (A + tB)) = X \quad \forall\, t \in [0,1], \|t - t_0\| \le \delta$$

The result for $t \in [0,1]$ follows by applying the local result for $t_0 \pm \delta$, $t_0 \pm 2\delta, \ldots$

$$\left.\begin{array}{l} A + t_0 B \text{ dissipative} \Rightarrow \|(I - (A + t_0 B))x\| \ge \|x\| \\ \text{Range}(I - (A + t_0 B)) = X \end{array}\right\} \Rightarrow$$

$(I - (A + t_0 B))^{-1} = R(t_0)$ exists and $\|R(t_0)\| \le 1$

Next we show that $BR(t_0)$ is bounded:

$$\|Bx\| \le \alpha \|Ax\| + \beta \|x\| \le \alpha \|(A + t_0 B)x\| + \alpha t_0 \|Bx\| + \beta \|x\|$$

or $\|Bx\| \le \frac{\alpha}{1-\alpha} \|(A + t_0 B)x\| + \frac{\beta}{1-\alpha} \|x\|$

$$(A + t_0 B) R(t_0) = R(t_0) - I$$

$$\Rightarrow \|BR(t_0)x\| \le \frac{\alpha}{1-\alpha} \|R(t_0) - I\| \, \|x\| + \frac{\beta}{1-\alpha} \|R(t_0)\| \, \|x\|$$

or $\|BR(t_0)x\| \leq \dfrac{2\alpha+\beta}{1-\alpha}\|x\|$

Now

$$I-(A+tB) = I-(A+t_0B)+(t_0-t)B$$
$$= \underbrace{\left(I+(t_0-t)BR(t_0)\right)}_{\substack{\text{invertible} \\ \text{if } \delta \leq \frac{1}{2}\frac{1-\alpha}{2\alpha+\beta}}}\underbrace{\left(I-(A+t_0B)\right)}_{\text{invertible}}$$

So for $|t-t_0| \leq \delta \leq \frac{1}{2}\frac{1-\alpha}{2\alpha+\beta}$

$$\text{Range}(I-(A+tB)) = X.$$

This finishes the lemma and the proof of the Theorem QED.

<u>Remark</u> In general one cannot allow $\alpha=1$ in the previous theorem or Lemma. However if X is reflexive (or B^* has dense domain in X^*) then under the assumptions of the previous theorem with $\alpha=1$ we have that $\overline{A+B}$ is the infinitesimal generator of a semigroup of contractions.

3.4 The abstract Cauchy Problem (the bounded operator case)

Consider X a Banach space and

$$B : X \to X \text{ linear and bounded.}$$

(a) The homogeneous Cauchy problem

$$(1) \begin{cases} \dfrac{du}{dt} = Bu & t > 0 \\ \\ u(0) = x \in X \end{cases}$$

<u>Def</u> $u : [0,T) \to X$ continuous, continuously differentiable on $(0,T)$ and satisfying (1) is called a classical solution of (1).

<u>Lemma</u> (integral form) The following are equivalent:

(i) u is a classical solution of (1) on $[0,T)$

(ii) $u : [0,T) \to X$ is continuous and satisfies

$$(2) \quad u(t) = x + \int_0^t Bu(s)\, ds \quad \forall\ 0 \le t < T.$$

<u>Proof</u> (i) $\Rightarrow$ (ii) by integration (Riemann integrals on Banach spaces)

(ii) $\Rightarrow$ (i).

$\mathbb{B} : X \to X$ is continuous

$\Rightarrow B U : [0, T) \to X$ is continuous

$\Rightarrow$
$$t \to \int_0^t B U(s)\, ds \quad \text{differentiable on } [0, T)$$

(by properties of Riemann integrals)

Differentiating (2) we get

$$\frac{dU}{dt} = B U(t) \qquad t \geq 0$$

Since right hand side is continuous on $[0, T)$ we get that U is a classical solution of (1).

<u>Remarks</u> 1° (ii) $\Rightarrow$ (i) is not valid in general for unbounded operators B!

2° Proof of (ii) $\Rightarrow$ (i) shows that classical solutions are actually continuously differentiable on $[0, T)$.

Theorem: (1) has a unique classical solution given by the uniform continuous semigroup of bounded operators generated by B:

$$U(t) = e^{tB} x \qquad t \geq 0$$

Proof: Uniqueness: Let $U_1 : [0, T_1) \to X$
$U_2 : [0, T_2) \to X$
be two classical solutions. We show that

$$U_1 \equiv U_2 \text{ on } [0, T_1) \cap [0, T_2).$$

WLOG we can assume $T_1 \leq T_2$

Claim for any $\tilde{T} < T$ (2) has a unique solution in

$$C([0, \tilde{T}], X) = \{ v : [0, \tilde{T}] \to X \mid v \text{ continuous} \}.$$

Indeed endow $C([0, \tilde{T}], X)$ with the Bielecki norm:

$$\|v\|_\sigma = \sup_{0 \leq t \leq T} e^{-\sigma t} \|v(t)\|$$

with $\delta > \|B\|$ fixed. Then

$$\left(C([0,\tilde{T}],X), \ \| \ \|_\delta \right) \text{ is a Banach}$$

space (with the norm equivalent with the standard supremum norm)

Let $K : C([0,\tilde{T}),X) \to C([0,\tilde{T}),X)$

$$(Kv)(t) = \int_0^t B\, v(s)\, ds$$

then K is well defined and a contraction with Lipschitz constant $\|B\|/\delta$:

$$\| Kv_1 - Kv_2 \|_\delta \leq \frac{\|B\|}{\delta} \|v_1 - v_2\|_\delta .$$

(2) is equivalent with

$$v = x + Kv$$

which by Contraction Principle has a unique solution in the space of continuous functions. The claim is proven.

Now U_1, U_2 are continuous sln's of (2) on $[0,T_1)$, hence on $[0,\tilde{T}]$ for all $\tilde{T} < T_1$.
So they must coincide with the unique sln of (2) on $[0,\tilde{T}]$. Hence

$$U_1(t) = U_2(t) \quad \forall\ 0 \le t \le \tilde{T},\ \forall \tilde{T} < T_1$$

$\Rightarrow$ $U_1 \equiv U_2$ on $[0,T_1)$. Uniqueness is proven!

<u>Existence</u> $U(t) = e^{tB}x$ is a classical solution of (1) on $[0,\infty)$ follows from the properties of e^{tB}, see Lecture 11. QED

(ii) <u>The inhomogeneous problem</u>

Consider

$$(3) \begin{cases} \dfrac{dU}{dt} = BU + f(t) & t > 0 \\ U(0) = x \in X \end{cases}$$

where $B: X \to X$ is bounded and $f: (0,T) \to X$.
The definition of classical solutions is as before

Theorem (i) If u is a classical solution of (3) and $f \in L^1[(0,T), X]$ then u satisfies:

(4) $$u(t) = e^{tB}x + \int_0^t e^{(t-s)B} f(s)\,ds \quad 0 \le t \le T$$

(ii) If $f \in L^1[(0,T), X] \cap C[(0,T), X]$ then (4) is the classical solution of (3).

Proof (i) Consider fix $0 \le t \le T$. Consider

$$g : [0,t) \to X \; ; \; g(s) = e^{(t-s)B} u(s)$$

Since $u(s)$ is a classical solution hence differentiable on $(0,t)$ we have for $0 < s < t$:

$$\frac{dg}{ds} = -e^{(t-s)}B\,u(s) + e^{(t-s)B} u'(s)$$

$$= e^{(t-s)}B\,u(s) + e^{(t-s)B}[Bu(s) + f(s)]$$

$$= \underbrace{e^{(t-s)B} f(s)}_{\in L^1(0,T).}$$

integrating on $(0,T)$ we get

$$g(t) - g(0) = \int_0^t e^{(t-s)B} f(s)\,ds$$

$$\Rightarrow \quad u(t) - e^{tB}x = \int_0^t e^{(t-s)B} f(s)\,ds$$

(ii) By (i) the classical solution is unique. $u(t)$ given by (4) is evidently continuous on $[0,T)$ and $u(0) = x$.

Since f is continuous on $(0,T)$

$\Rightarrow t \to \int_0^t e^{(t-s)B} f(s)\,ds$ is differentiable on $(0,T)$

and $$\frac{d}{dt}\int_0^t e^{(t-s)B} f(s)\,ds = \underbrace{f(t) + B\int_0^t e^{(t-s)B} f(s)\,ds}_{\text{continuous on } (0,T)}$$

Hence $u(t)$ given by (4) is continuously differentiable on $(0,T)$ and

$$\frac{du}{dt} = Be^{tB}x + f(t) + B\int_0^t e^{(t-s)B} f(s)\,ds$$

$$= B\left(e^{tB}x + \int_0^t e^{(t-s)B} f(s)\,ds\right) + f(t)$$

$$= B\,u(t) + f(t) \qquad \text{QED}$$

Remark If $f(t)$ is not continuous on $(0,T)$. (3) has no classical solutions.

Def If $f \in L^1[(0,T), X]$ then v given by (4) is called a mild solution

Theorem (i) If v is a mild solution of (3) then $\exists \{f_n\} \subseteq L^1[(0,T), X]$ $f_n \xrightarrow{L^1} f$ and $\{x_n\} \subseteq X$ $x_n \to x$ such that

$$\begin{cases} \dfrac{dv_n}{dt} = Bv_n + f_n(t) \\ v_n(0) = x_n \end{cases}$$

has classical solutions and $v_n \longrightarrow v$ in $C([0,T], X)$.

(ii) If v is a mild solution of (3) then $\frac{dv}{dt}$ exists almost everywhere $\frac{dv}{dt} \in L^1[(0,T), X]$ and $\frac{dv}{dt} = Bv + f(t)$ a.e. (in other words v is a strong solution of (3)).

Proof (i) $f \in L^1[(0,T),X]$

$\Rightarrow \exists \{f_n\} \subseteq C([0,T],X)$ such that

$f_n \xrightarrow{L^1} f$ (i.e.:

$$\lim_{n\to\infty} \int_0^T \|f(t) - f_n(t)\| \, dt = 0.)$$

Then by previous theorem part (ii):

$$\begin{cases} \dfrac{du_n}{dt} = B u_n + f_n(t) \\ u_n(0) = x \end{cases}$$

has classical solution u_n given by:

$$u_n(t) = e^{tB} x + \int_0^t e^{(t-s)B} f_n(s) \, ds$$

Then

$$|u_n(t) - u(t)| \le \int_0^t e^{(t-s)\|B\|} \|f_n(s) - f(s)\| \, ds.$$

$$\le e^{T\|B\|} \|f_n - f\|_{L^1[(0,T),X]} \quad \text{QED}$$

(ii) $e^{tB}\int_0^t \underbrace{e^{-sB} f(s)}_{\in L^1((0,T),X)} ds$ is differentiable a.e.

with derivative:

$$B e^{tB}\int_0^t e^{-sB} f(s)\, ds + e^{tB} e^{-tB} f(t) \quad \text{a.e.}$$

Since e^{tB} is differentiable everywhere $\Rightarrow$ v given by (4) is differentiable a.e.:

$$\frac{dv}{dt} = B e^{tB} x + B e^{tB}\int_0^t e^{-sB} f(s)\, ds + f(t)$$

$$= B v(t) + f(t) \quad \text{a.e.} \qquad \text{QED.}$$