Lecture 14 Math 555

Note Title 4/14/2008

Summary

3.4 The abstract Cauchy problem (the case of unbounded operators)

3.5 Nonlinear evolution equations

3.4 The abstract Cauchy problem

Let X be a Banach space and

$A : D(A) \to X$ a linear operator

(i) The homogeneous Cauchy problem:

$$(1)\begin{cases} \dfrac{du}{dt} = Au & t > 0 \\ u(0) = x \in X \end{cases}$$

Def $u : [0,T) \to X$ continuous, continuous differentiable on $(0,T)$ with $u(t) \in D(A)$ on $(0,T)$ satisfying (1) is called a classical solution of (1).

Theorem (uniqueness) Assume $\overline{D(A)} = X$ and $R(\lambda, A)$ exists for all $\lambda \geq \lambda_0$ with

$$\sup_{\lambda \to \infty} \frac{\log \| R(\lambda, A) \|}{\lambda} = 0$$

then (1) can have at most one classical solution for any $x \in X$

Proof See Pazy Ch 4 Theorem 1.2.

Theorem (connection with semigroups of operators)

(a) If A is the infinitesimal generator of a C_0 semigroup of bounded operators $T(t), t \geq 0$ then (1) has a unique classical solution for all $x \in D(A)$ given by:

$$u(t) = T(t) x$$

(b) If (1) has a unique classical solution for all $x \in D(A)$ and all this solutions are continuously differentiable on $[0, T)$ for some $T > 0$ then A is the infinitesimal generator of a C_0 semigroup of operators.

<u>Proof</u> (a) $T(t)x$, $x \in D(A)$ is a classical solution follows from the properties of C_0 semigroups of operators. Uniqueness follows from previous theorem since A infinitesimal generator of $T(t)$ with $\|T(t)\| \leq M e^{t\omega}$ implies:

$R(\lambda, A)$ exists for $\lambda > \omega$ and

$$\|R(\lambda, A)\| \leq \frac{M}{\lambda - \omega}$$

$$\Rightarrow \|R(\lambda, A)\|^{1/\lambda} \leq \frac{M^{1/\lambda}}{(\lambda-\omega)^{1/\lambda}} \longrightarrow 1 \text{ as } \lambda \to \infty$$

$$\Rightarrow \frac{\log \|R(\lambda, A)\|}{\lambda} \longrightarrow 0 \text{ as } \lambda \to \infty$$

(b) See Pazy Chapter 4 Theorem 1.3.

<u>Def</u> If A is the infinitesimal generator of a C_0 semigroup of operators then

$U(t) = T(t)x$, $x \in X \setminus D(A)$ is called a mild solution of (1).

<u>Remark</u> For any mild solution $u(t)$ of (1) there exists a sequence of classical solutions $u_n(t)$ such that

$$u_n(t) \to u(t) \quad \text{uniformly on any } [0,T].$$

Indeed for $x \in X \setminus D(A)$ consider $\{x_n\} \subset D(A)$ such that $x_n \to x$ then

$$u_n(t) = T(t)x_n \text{ are classical solutions and}$$
$$\|T(t)x - T(t)x_n\| \leq Me^{\omega T}\|x - x_n\| \text{ on } 0 \leq t \leq T.$$

<u>(ii) The inhomogeneous Cauchy problem</u>

Let $f : (0,T) \to X$

$$(2) \begin{cases} \dfrac{du}{dt} = Au + f(t) & 0 < t < T \\ u(0) = x \in X \end{cases}$$

<u>Def</u> 1° Classical solution as before

2° Strong solution: $u : [0,T) \to X$ continuous $\frac{du}{dt} \in L^1[(0,T), X]$, $u(t) \in D(A)$ a.e. on $(0,T)$ and

u satisfies (2) a.e. on $(0,T)$.

3° Mild solution: If A is the infinitesimal generator of a C_0 semigroup of operators $T(t), t \geq 0$ $f \in L^1[(0,T), X]$ then

$$U(t) = T(t)x + \int_0^t T(t-s) f(s)\, ds \quad 0 \leq t \leq T$$

is called the mild solution of (2).

<u>Remark</u> The mild solution is continuous on $[0,T]$.

<u>Theorem 1</u> If A is the infinitesimal generator of a C_0 semigroup of operators $T(t)$, $t \geq 0$ and $f \in L^1[(0,T), X]$ then any classical or strong solution of (2) is also a mild solution of (2).

<u>Proof</u> Let $U(t)$ be a strong solution of (2). Fix $t > 0$ and denote:

$$g(s) = T(t-s)U(s)$$

then g is differentiable a.e. on $0 < s < t$ and

$$g'(s) = -T(t-s)AU(s) + T(t-s)U'(s)$$
$$= \underbrace{T(t-s)f(s)}_{L^1[(0,t), X]}$$

Hence by integrating on $[0,t]$ we get

$$g(t) - g(0) = \int_0^t T(t-s) f(s)\, ds \quad \text{or.}$$

$$U(t) - T(t)x = \int_0^t T(t-s) f(s)\, ds \qquad \text{QED.}$$

Reciprocals: Let $v(t) = \int_0^t T(t-s) f(s)\, ds$

Theorem 2 If A is infinitesimal generator of a C_0 semigroup $f \in L^1[(0,T), X]$ and f continuous on $(0,T)$ then (2) has a classical solution on $[0,T)$ $\forall x \in D(A)$ iff

(i) $v(t)$ is continuously differentiable on $(0,T)$;

or equivalently.

(ii) $v(t) \in D(A)$ $\forall\, 0<t<T$ and $Av(t)$ is continuous on $(0,T)$

Proof Classical solutions $\Rightarrow$ (i)

$$v(t) = \underbrace{U(t) - T(t)x}_{\text{continuously differentiable on } (0,T) \text{ for } x \in D(A)} \Rightarrow v'(t) = U'(t) - T(t)Ax$$

(i) $\Rightarrow$ (ii) For $h>0$ and $0<t<T$ we have:

$$(3) \quad \frac{T(h) - Id}{h} v(t) = \frac{v(t+h) - v(t)}{h} - \frac{1}{h}\int_t^{t+h} T(t+h-s) f(s)\, ds$$

From f continuous on $[t, t+h]$ we have RHS has a limit when $h \to 0 \Rightarrow v(t) \in D(A)$ and

$$Av(t) = v'(t) - f(t)$$

$\Rightarrow Av(t)$ continuous on $(0,T)$

(ii) $\Rightarrow$ classical solutions

Now LHS in (3) and integral term in RHS have limits as $h \to 0$ we have

$$Av(t) = D^+ v(t) - f(t) \text{ or } D^+ v(t) = Av(t) + f(t)$$

hence $D^+ v(t)$ is continuous on $(0,T) \Rightarrow v'$ exists on $(0,T)$ is continuous and

$$v'(t) = Av(t) + f(t)$$

hence $u(t) = T(t)x + v(t)$ is continuously differentiable on $(0,T)$ for $x \in D(A)$ and

$$u'(t) = AT(t)x + Av(t) + f(t) = Au(t) + f(t)$$

Therefore $u(t) = T(t)x + v(t)$ is a classical sln QED

<u>Corollary 2</u> If A infinitesimal generator of a C_0 semigroup $f \in L^1[(0,T), X]$, f continuous on $(0,T)$ and

(i) f is continuously differentiable on $[0,T)$; or

(ii) $f(s) \in D(A)$ for $0 < s < T$ and $Af(s) \in L^1[(0,T), X]$;

then (2) has a unique classical solution for any $x \in D(A)$.

Proof (i) $v(t) = \int_0^t T(t-s) f(s)\, ds = \int_0^t T(s) f(t-s)\, ds$

$\Rightarrow v'(t) = T(t) f(0) + \int_0^t T(s) f'(t-s)\, ds.$

$\Rightarrow v$ continuously differentiable on $(0,T)$

(ii) By A closed:

$$A v(t) = A \int_0^t T(t-s) f(s)\, ds = \int_0^t A T(t-s) f(s)\, ds$$

$$= \int_0^t T(t-s) A f(s)\, ds \text{ continuous on } (0,T)$$

Remark If u is the mild solution of (2) then u is the uniform limit on $[0,T]$ of classical solutions of (2).

Indeed let $\{f_n\} \subset C^1([0,T], X)$ such that

$$\int_0^T \| f_n(t) - f(t) \| \, dt \to 0 \text{ as } n \to \infty.$$

and $\{x_n\} \subseteq D(A)$ such that $x_n \to x$. Then

$$\begin{cases} \dfrac{du_n}{dt} = A u_n + f_n(t) \\ u_n(0) = x_n \end{cases}$$

has classical solutions $u_n(t) = T(t) x_n + \int_0^t T(t-s) f_n(s)\, ds$

and $v_n \rightrightarrows v$ on $[0, T]$

<u>Theorem 3</u> If A is infinitesimal generator of a C_0 semigroup, $f \in L^1[(0,T), X]$ then (2) has a strong solution on $[0,T]$, $\forall x \in D(A)$ iff

(i) $v(t)$ is differentiable a.e. on $(0,T)$ and $v \in L^1[(0,T), X]$,

or equivalently:

(ii) $v(t) \in D(A)$ a.e. on $[0,T)$ and $Av(t) \in L^1[(0,T), X]$

<u>Proof</u> Adapt the one for Theorem 2.

<u>Corollary 3</u> If $f' \in L^1[(0,T), X]$ or f is Lipschitz continuous on $[0,T]$ and X is reflexive, then (2) has a unique strong solution for any $x \in D(A)$.

3.5 Nonlinear evolution equations.

Let X be a Banach space:

$$(4) \begin{cases} \dfrac{du}{dt} = Au + f(t, u(t)) & t > t_0 \\ u(t_0) = u_0 \in X \end{cases}$$

where $A : D(A) \to X$ is the infinitesimal generator

of a C_0 semigroup of operators and
$f : [t_0, T) \times X \to X$

Lemma If u is a classical or strong solution of (4) and $f : [t_0, T] \times X \to X$ is continuous then u satisfies:

$$u(t) = T(t-t_0)x + \int_{t_0}^{t} T(t-s) f(s, u(s))\, ds \tag{5}$$

Proof As before, fix $t > t_0$ and denote

$$g(s) = T(t-s) u(s)$$

Then g is differentiable a.e. on $t_0 < s < t$ and

$$g'(s) = -T(t-s) A u(s) + T(t-s) u'(s)$$
$$= \underbrace{T(t-s) f(s, u(s))}_{\text{continuous on } [t_0, t]}$$

By integrating.
$$g(t) - g(t_0) = \int_{t_0}^{t} T(t-s) f(s, u(s))\, ds.$$

$$\Rightarrow u(t) = T(t-t_0) u_0 + \int_{t_0}^{t} T(t-s) f(s, u(s))\, ds \quad \text{QED.}$$

<u>Def</u> A continuous solution of (5) is called a mild solution of (4).

<u>Theorem</u> (existence of mild solutions) If A is the infinitesimal generator of a C_0 semigroup of operators and $f : [t_0, T] \times X \to X$ is continuous in t and

$$\| f(t, u) - f(t, v) \| \leq L \| u - v \| \quad \forall t \in [t_0, T]$$
$$\forall u, v \in X$$

then problem (4) has a unique mild solution that depends continuously on the initial data u_0.

<u>Proof</u> Consider

$$K : C([t_0, T] \times X, X) \to C([t_0, T] \times X, X)$$

$$(Kv)(t) := \int_{t_0}^{t} T(t-s) f(s, v(s)) \, ds$$

continuity of $f \Rightarrow K$ well defined.
f Lipschitz $\Rightarrow K$ Lipschitz.

To make K contractive use Bielecki norms:

$$\| v \|_\delta = \sup_{t_0 \leq t \leq T} e^{-\delta (t - t_0)} \| v(t) \| \quad \text{with } \delta > \omega + ML$$

$$\|(Kv_1 - Kv_2)(t)\| \le \int_{t_0}^{t} \|T(t-s)\| \, \|f(s, v_1(s)) - \|f(s, v_2(s))\|$$

$$\le \int_{t_0}^{t} M e^{\omega(t-s)} L \|v_1(s) - v_2(s)\| \, ds$$

$$\le \|v_1 - v_2\|_\sigma \int_{t_0}^{t} M e^{\omega(t-s)} e^{\sigma(s-t_0)} \, ds$$

$$= ML \, e^{\omega t} \frac{e^{(\sigma-\omega)s}}{\sigma-\omega} \Big|_{t_0}^{t} e^{-\sigma t_0} \quad \|v_1 - v_2\|_\sigma$$

$$\Rightarrow \sup_{t_0 \le t \le T} e^{-\sigma(t-t_0)} \|(Kv_1 - Kv_2)(t)\|$$

$$\le \frac{ML}{\sigma-\omega} \left(1 - e^{-(\sigma-\omega)(t-t_0)}\right) \|v_1 - v_2\|_\sigma$$

$$\Rightarrow \|Kv_1 - Kv_2\|_\sigma \le \frac{ML}{\sigma-\omega} \|v_1 - v_2\|_\sigma$$

with $ML/(\sigma-\omega) < 1$.

$\Rightarrow$ $u(t) = T(t-t_0)u_0 + (Ku)(t)$ has a unique solution in $C([t_0, T], X)$ for any $u_0 \in X$.
Moreover if v solves $v(t) = T(t-t_0)v_0 + Kv(t)$ we have

$$\|u - v\|_\sigma \le M\|u_0 - v_0\| + \frac{ML}{\sigma-\omega} \|u - v\|_\sigma$$

$$\Rightarrow \|u - v\|_\sigma \le \frac{M}{1 - (ML)/(\sigma-\omega)} \|u_0 - v_0\|$$

hence $\sup_{t_0 \le t \le T} \|u(t) - v(t)\| \le \dfrac{M e^{\sigma(T-t_0)}}{1 - \frac{ML}{\sigma - \omega}} \|u_0 - v_0\|$

$\Rightarrow$ the mild solution depends continuously (Lipschitz) on the initial data. QED

<u>Theorem</u> (local existence and maximal solutions)
Let A be the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$. Assume $f: [0,T) \times X \to X$ satisfies:

(a) $f(\cdot, x): [0,T) \to X$ is continuous $\forall x \in X$

(b) $\forall t_1, t_2 \in [0,T)$, $\forall M > 0$ $\exists L(t_1, t_2, M)$ such that for all $t \in [t_1, t_2]$ and $u, v \in X$ with $\|u\|, \|v\| \le M$:

$$\|f(t,u) - f(t,v)\| \le L(t_1, t_2, M) \|u - v\|$$

then for any $x \in X$ the problem

$$\begin{cases} \dfrac{du}{dt} = Au + f(t,u) & 0 < t < T \\ u(0) = x \end{cases}$$

has a unique mild solution $u: [0, T_{max}) \to X$ such that either

(i) $T_{max} = T$

or (ii) $\lim_{t \to T_{max}} \|u(t)\| = +\infty$

<u>Proof</u> First we show "local existence":
$\forall t_0 \in [0,T)$, $\forall x \in X$ $\exists T(t_0, \|v_0\|) \in (t_0, T)$ such that

$$(5) \quad u(t) = T(t-t_0)v_0 + \int_{t_0}^{t} T(t-s) f(s, u(s))\, ds$$

has a unique solution in $C([t_0, T(t_0, \|x\|)], X)$.

Recall that $\|T(t)\| \leq M e^{\omega t}$. Choose $\delta > 0$ such that

$$e^{\omega \delta} \leq 2$$

Fix $t_0 \in [0,T)$, $v_0 \in X$ and let $t_1 = \min\{t_0 + \delta, \frac{t_0+T}{2}\}$

$$K : C([t_0, t_1], X) \to C([t_0, t_1], X).$$

$$(Kv)(t) = T(t-t_0)v_0 + \int_{t_0}^{t} T(t-s) f(s, v(s))\, ds$$

Now we seek $t_0 < T_1 \leq t_1$ such that:

$$K : C([t_0, T_1], B(0,R)) \to C([t_0, T_1], B(0,R))$$

$$R = 4M\|v_0\|$$

where $B(0,R) = \{x \in X \mid \|x\| \leq R\}$.
In other words K leaves the ball of radius R invariant in the Banach space $C([t_0, T_1], X)$ endowed with supremum norm.

We use that for $t_0 \leq s \leq t_1$:

$$\| T(t-s) \| \leq M e^{\omega(t-s)} \leq M e^{\omega \delta} \leq 2M$$

$$\| f(s,v) - f(s,0) \| \leq L(t_1, t_2, R) R \quad \text{for } v \in B(0,R)$$

to get:

$$\| (Kv)(t) \| \leq 2M \|v_0\| + \int_{t_0}^{t} 2M L R \, ds + \int_{t_0}^{t} 2M \| f(s,0) \| \, ds$$

$$\leq 2M \|v_0\| + R \, 2ML(t-t_0) + 2MF(t-t_0)$$

where $F = \sup_{t_0 \leq s \leq t_1} \| f(s,0) \|$

To have $\| (Kv)(t) \| \leq R = 4M \|v_0\|$ for $t_0 \leq t \leq T_1$ it suffices to choose $t_0 < T_1 \leq t_1$ such that

$$2M\|v_0\| + 4M\|v_0\| \, 2ML(T-t_0) + 2MF(T-T_0) \leq 4M\|v_0\|$$

Hence

$$T_1 = \min \left\{ t_1, \; t_0 + \frac{2M\|v_0\|}{8M^2 L \|v_0\| + 2MF} \right\}.$$

With this choice

$$K : C([t_0, T_1], B(0,R)) \to C([t_0, T_1], B(0,R))$$

is well defined. Moreover for $t_0 \leq t \leq T_1$:

$$\|(Kv_1 - Kv_2)(t)\| \leq \int_{t_0}^{t} 2ML\, \|v_1(s) - v_2(s)\|\, ds$$

$$\leq \frac{4M^2L\|v_0\|}{8M^2L\|v_0\| + 2MF} \sup_{t_0 \leq s \leq T_1} \|v_1(s) - v_2(s)\|$$

$$\leq \frac{1}{2} \sup_{t_0 \leq s \leq T_1} \|v_1(s) - v_2(s)\|$$

Hence K is a contraction with Lipschitz constant $\frac{1}{2}$ in the complete metric space

$$C([t_0, T_1], B(0,R))$$

endowed with the supremum metric:

$$d_\infty(v_1, v_2) = \sup_{t_0 \leq s \leq T_1} \|v_1(s) - v_2(s)\|_X$$

(Note that $B(0,R) = \{x \in X : \|x\| \leq R\}$ is closed).
Consequently K has a unique fixed point which is equivalent to (5) has a unique sln:

$$v \in C([t_0, T_1], B(0,R)).$$

Consider now another solution of (5)

$$\tilde{v} \in C([t_0, T_1], X)$$

and let

$$J = \{t \in [t_0, T_1] \mid v(s) = \tilde{v}(s) \text{ for } t_0 \leq s \leq t\}$$

then $t_0 \in J$ since $v(t_0) = \tilde{v}(t_0) = v_0$.

J is closed since both $v, \tilde{v}$ are continuous

J is open in $[t_0, T_1]$: if $t_2 \in J$, $t_2 < T_1$ then from the choice of T_1 we get $\|u(t_2)\| < R \Rightarrow$ $\|u(t_2)\| = \|\tilde{u}(t_2)\| < R$ and from continuity of $\tilde{u}$ $\Rightarrow \exists\, T_2 > t_2$ such that $\|\tilde{u}(t)\| \leq R$ on $t_0 \leq t \leq T_2$

Since $K\big|_{C([t_0,T_2],\, B(0,R))}$ is contractive

we get $u(t) = \tilde{u}(t)$ on $[t_0, T_2]$ so

$[t_0, t_2 + T_2 - t_2) \subseteq J \Rightarrow J$ open

In conclusion $u(t) = \tilde{u}(t)$ on $[t_0, T_1]$ and (5) has a unique solution in $C([t_0,T_1], X)$.

<u>Maximal solution</u>: Consider

$$(6) \qquad u(t) = T(t)x + \int_0^t T(t-s)\, f(s, u(s))\, ds \qquad 0 \leq t < T$$

Let $S = \{ u : [0, \tau) \longrightarrow X \mid u \text{ continuous},\ u \text{ satisfies (6) on } [0,\tau) \}$

$S \neq \emptyset$: Use the local existence for $t_0 = 0$ and choose $\tau = T(0, \|x\|)$.

Define the order relation on S

$u_1 \leq u_2$ iff u_2 is defined on a larger interval than u_1 and $u_1(t) = u_2(t)$ on their common interval of definition.

Consider a totally ordered subset $C \subseteq S$ indexed by J

$$C = \{ v_j : I_j \to X \mid j \in J,\ v_{j_1} \le v_{j_2} \text{ or } v_{j_2} \le v_{j_1} \ \forall j_1, j_2 \in J \}$$

then define

$$v_C : \bigcup_{j \in J} I_j \longrightarrow X$$

by $v_C(t) = v_j(t)$ if $t \in I_j$

<u>Note</u> v_C is well defined: for $t \in I_{j_1} \cap I_{j_2}$ we have $v_{j_1} \le v_{j_2}$ or $v_{j_2} \le v_{j_1}$ in any case $v_{j_1}(t) = v_{j_2}(t)$

v_C is continuous, since for $t \in I_j$ $\exists\, \varepsilon < 0$ such that $(t-\varepsilon, t+\varepsilon) \subseteq I_j \Rightarrow v_C = v_j$ on $(t-\varepsilon, t+\varepsilon) \Rightarrow v_C$ continuous.

$v_j \le v_C \quad \forall j \in J$.

So v_C is an upper bound for C. By Zorn's Lemma S has a maximal element

$v : [0, T_{max}) \to X$ continuous, v satisfies (6) on $[0, T_{max})$, hence v is a mild solution of the problem. If $\tilde{v} : [0, \tilde{T}_{max}) \to X$ is another maximal element then WLOG consider $\tilde{T}_{max} \le T_{max}$ and

$$J = \{ t \in [0, \tilde{T}_{max}) : v(s) = \tilde{v}(s) \text{ on } 0 \le s \le t \}$$

then $0 \in J$, J is closed by continuity of v and $\tilde{v}$, and J is open

Since for $t_0 \in Y$ $\exists\, T(t_0, \|v(t_0)\|) \in [t_0, T_{max})$ such that (5) has a unique solution in $C([t_0, T(t_0, \|v_0\|)], X)$ but both v and $\tilde{v}$ are solutions of (5) in $C([t_0, T(t_0, \|v_0\|)], X)$ hence $[0, T(t_0, \|v_0\|)) \subseteq Y \Rightarrow Y$ open.

So $\tilde{v}(t) = v(t)$ on $[0, \tilde{T}_{max}) \Rightarrow \tilde{v} \le v \overset{\text{maximality}}{\Rightarrow} \tilde{v} = v$.

Finally if $T_{max} < T$ and $\lim\limits_{t \to T_{max}} \|v(t)\| < \infty$

then consider $t_n \nearrow T_{max}$ and

$$N = \sup_{n \in \mathbb{N}} \{\|v(t_n)\|\} < \infty$$

Note that

$$(5_n) \quad v(t) = T(t - t_n) v(t_n) + \int_{t_n}^{t} T(t-s) f(s, v(s))\, ds.$$

But by local existence (5_n) has unique solutions in

$$C([t_n, T(t_n, \|v(t_n)\|)], X)$$

and

$$T(t_n, \|v(t_n)\|) - t_n \geq T(t_1, N) - t_1 > 0$$

$\Rightarrow$ v can be extended on

$$[0, T_{max} + T(t_1, N) - t_1)$$

in contradiction with the maximality of v.