Summary:

3.5. Nonlinear ev. eq (continuation)
3.6 Applications to a nonlinear Schrödinger eq

3.5 Nonlinear evolution equations (cont)

$$(1)\begin{cases}\frac{du}{dt} = Au + f(t, u(t)) & t > t_0 \\ u(t_0) = u_0 \in X\end{cases}$$

X a Banach space $A: D(A) \to X$ linear
$f: (t_0, T) \times X \to X$. Recall:

<u>Def</u> (classical solution) $u: [t_0, t_1) \to X$ continuous, continuously differentiable on (t_0, t_1), $u(t) \in D(A)$ for all $t \in (t_0, t_1)$ and satisfying (1) is called a classical solution.

(strong solution) $u: [t_0, t_1) \to X$ continuous, $\frac{du}{dt} \in L^1[(t_0, t_1), X]$, $u(t) \in D(A)$ a.e. on (t_0, t_1) and satisfying the eq in (1) a.e. and $u(t_0) = u_0$ is called a strong solution (mild solution) If A is the infinitesimal generator of a C_0 semigroup of bounded operators $T(t)$, $t \geq 0$, then any $u \in C([t_0, t_1), X)$ satisfying

$$(2) \quad u(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s) f(s, u(s))\, ds$$

for $t_0 \leq t < t_1$ is called a mild solution.

<u>Lemma</u> If A is the infinitesimal generator of a C_0 semigroup and $f: [t_0, T) \times X \to X$ is continuous then any classical or strong solution of (1) are also mild solutions

<u>Theorem 1</u> If A is the infinitesimal generator of a C_0 semigroup and $f: [t_0, T] \times X \to X$ satisfies

(i) $f(\cdot, x)$ is continuous $\forall x \in X$

(ii) $\exists L > 0$ such that $\forall t \in [t_0, T], \forall x, y \in X$

$\| f(t,x) - f(t,y) \| \leq L \| x - y \|$

then (2) has a unique (mild) solution in $C([t_0, T], X)$.

<u>Theorem 2</u> If A is the infinitesimal generator of a C_0 semigroup and $f: [0, T) \times X \to X$ satisfies (i) above and

(ii)' $\forall\, t_0, t_1 \in [0,T),\ R>0\ \exists\, L(t_0,t_1,R)>0$ such that for all $t \in [t_0,t_1]$ and $x, y \in X$ $\|x\|, \|y\| \le R$

$$\|f(t,x) - f(t,y)\| \le L(t_0,t_1,R)\, \|x-y\|$$

then (2) has a unique maximal solution:

$$U : [0, T_{max}) \longrightarrow X \text{ continuous}$$

such that either $T_{max} = T$ or $\lim\limits_{t \to T_{max}} \|U(t)\| = +\infty$

<u>Regular solutions of (1):</u>

<u>Theorem 3</u> Let A be the infinitesimal generator of a C_0 semigroup $T(t)$, $t \ge 0$, and $(D(A), |\cdot|_A)$ be the Banach space with norm

$$|x|_A = \|x\| + \|Ax\|.$$

Assume that $f : [t_0, T] \times D(A) \longrightarrow D(A)$ satisfies:

(i) $f(\cdot, x)$ is continuous $\forall\, x \in D(A)$

(ii) $\exists\, L>0$ such that $\forall\, t \in [t_0,T]$, $x, y \in D(A)$

$$|f(t,x) - f(t,y)|_A \le L\, |x-y|_A$$

Then (1) has a unique classical solution on $[t_0, T]$ for $v_0 \in D(A)$.

Proof The proof of Theorem 1, see existence of mild solutions in lecture 14, page 11, carries through with $(X, \| \|)$ replaced by $(D(A), | |_A)$ and shows the existence of a unique $C([t_0,T], D(A))$ solution of

$$U(t) = T(t-t_0) U_0 + \int_{t_0}^{t} T(t-s) f(s, U(s))\, ds.$$

Let $g(s) = f(s, U(s))$ clearly $g : [t_0, T] \to D(A)$ is continuous (since $f : [t_0,T] \times D(A) \to D(A)$ and $U : [t_0, T] \to D(A)$ are both continuous). In particular

$$\begin{cases} g(s) \in D(A) \ \forall t \in [t_0, T] \text{ and} \\ Ag(s) : [t_0, T] \to X \text{ is continuous.} \end{cases}$$

(See Corollary 2, lecture 14 page 7)

$$\Rightarrow v(t) = T(t-t_0) U_0 + \int_{t_0}^{t} T(t-s) f(s, U(s))\, ds$$

is a classical solution of:

$$\begin{cases} \dfrac{dv}{dt} = Av + f(t, U(t)) \\ v(t_0) = U_0 \end{cases}$$

$\Rightarrow U(t) = v(t)$ is a classical solution of (1).

Theorem 4 Let A be the infinitesimal generator of a C_0 semigroup and $(D(A), |\cdot|_A)$ the Banach space with

$$|x|_A = \|x\| + \|Ax\|$$

Assume $f : [0,T) \times D(A) \to D(A)$ satisfies:

(i) $f(\cdot, x)$ is continuous $\forall x \in D(A)$

(ii)' $\forall t_1, t_2 \in [0,T), R > 0 \; \exists L(t_1,t_2,R) > 0$:

$$|f(t,x) - f(t,y)|_A \leq L(t_1,t_2,R)\, |x-y|_A$$

for all $t \in [t_1,t_2]$ and $x, y \in D(A)$ with $|x|_A, |y|_A \leq R$

Then for any $u_0 \in D(A)$ (1) has a unique classical maximal solutions

$$U : [0, T_{max}) \to D(A)$$

such that

$$T_{max} = T \text{ or } \lim_{t \to T_{max}} |U(t)|_A = \infty$$

Proof The proof of Th 2, see Lecture 14 page 13-19, carries over with $(X, \|\ \|)$ replaced by $(D(A), |\cdot|_A)$. The fact that the maximal (mild) solution is in fact classical follows as in Theorem 3 above.

Other ways to obtain regularity:

Theorem 5 Under the hypotheses of Theorem 1 if in addition $f: [t_0, T] \times X \to X$ is continuously (Frechet) differentiable then the mild solution of (1) is a classical solution for all $u_0 \in D(A)$.

Sketch of proof: Note that f continuously differentiable implies (i) and (ii) in Theorem 1. so we have a unique mild solution on $[t_0, T]$:

$$(2) \quad u(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s) f(s, u(s))\, ds$$

For $u_0 \in D(A)$, $T(t-t_0)u_0$ is continuously differentiable in t, for $t \geq t_0$. To show that the integral term is differentiable in t one would like to move the t dependence to f:

$$\frac{d}{dt} \int_{t_0}^{t} T(t-s) f(s, u(s))\, ds = \frac{d}{dt} \int_{0}^{t-t_0} T(s') f(t-s', u(t-s'))\, ds'$$

$$\underset{\text{formally}}{=} T(t-t_0) f(t_0, u(t_0)) + \int_0^{t-t_0} T(s') \frac{\partial f}{\partial s}(t-s', u(t-s'))\, ds'$$

$$+ \int_0^{t-t_0} T(s') \frac{\partial f}{\partial u}(t-s', u(t-s')) \frac{du}{dt}(t-s')\, ds'$$

$$= T(t-t_0) f(t_0, u(t_0)) + \int_{t_0}^{t} T(t-s) \frac{\partial f}{\partial s}(s, u(s))\, ds$$

$$+ \int_{t_0}^{t} T(t-s) \frac{\partial f}{\partial u}(s, u(s)) \frac{du}{dt}(s)\, ds$$

The problem is that we do not know whether $\frac{dv}{dt}$ even exists! To avoid this we use fixed point techniques to show that the equation for $\frac{dv}{dt}$ obtained by differentiating (2):

$$(*)\ w(t) = \overbrace{T(t-t_0)Av_0 + T(t-t_0)f(t_0, v(t_0))}^{\text{continuous in } t} +$$

$$+ \underbrace{\int_{t_0}^{t} T(t-s)\frac{\partial f}{\partial s}(s, v(s))\,ds}_{\text{continuous in } t} + \underbrace{\int_{t_0}^{t} T(t-s)\frac{\partial f}{\partial v}(s, v(s))\, w(s)\,ds}_{\text{uniformly Lipschitz in } v}$$

has a unique solution. Then we show that $w(t)$ is indeed $\frac{dv}{dt}$ by showing

$\lim_{h\to 0} \left\| \frac{v(t+h)-v(t)}{h} - w(t) \right\| = 0$. This follows from using (2) and (*) to estimate

$$w_h(t) = \frac{v(t+h)-v(t)}{h} - w(t). \text{ We get}$$

$$\|w_h(t)\| \le \varepsilon(h) + M\int_{t_0}^{t} \|w_h(s)\|\,ds.$$

where $\varepsilon(h) \to 0$ as $h \to 0$ and $M = \max_{t_0 \le s \le T} \{\|T(t-s)\frac{\partial f}{\partial v}(s, v(s))\|\}$.

By Gronwall $\|w_n(t)\| \leq \varepsilon(n) e^{(t-t_0)M}$
hence $w_n(t) \to 0$ as $n \to \infty$.

The last part, $v(t) \in D(A)$ $\forall\, t_0 < t < T$, and the fact that v satisfies (1) follows from

$$v(t) = T(t-t_0)v_0 + \int_{t_0}^{t} T(t-s) \underbrace{f(s, v(s))}_{} \, ds$$

$g(s)$ differentiable on $[t_0, T)$

is the classical solution of:

$$\begin{cases} \dfrac{dv}{dt} = Av + f(t, v(t)) \\ v(t_0) = v_0 \end{cases}$$

See Corollary 2, Lecture 14, page 7. Since $v = v$, v is a classical solution of (1). Q.E.D.

Theorem 6. Under the hypotheses of theorem 1 if X is reflexive and $f : [t_0, T] \times X \to X$ is Lipschitz (in both variables) then the unique mild solution is a strong solution of (1) for all $v_0 \in D(A)$.

Sketch of proof. From the formula for the mild solution (2) using that f is Lipschitz one gets:

$$\|u(t+h)-u(t)\| \le Ch + ML\int_{t_0}^{t}\|u(s+h)-u(s)\|\,ds$$

where $M = \max\{\|T(t)\| : t_0 \le t \le T\}$ and L is the Lipschitz constant for f. By Gronwall:

$$\|u(t+h)-u(t)\| \le Ch\, e^{ML(T-t_0)}$$

$\Rightarrow$ u is Lipschitz with constant $Ce^{ML(T-t_0)}$.

$\Rightarrow$ $s \mapsto f(s,u(s))$ is Lipschitz

$$\Rightarrow \quad v(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s) f(s,u(s))\,ds$$

is the strong solution of

$$\begin{cases} \dfrac{dv}{dt} = Av + f(t,u(t)) \\ v(t_0) = u_0 \in D(A) \end{cases}$$

See Corollary 3, lecture 14 page 8. Then $u(t) = v(t)$ is a strong solution of (1). QED.

3.5 Applications to a Nonlinear Schrödinger Eq

$$(3)\begin{cases} i\dfrac{\partial u}{\partial t} = -\Delta_x u + V(x)u + k|u|^2 u & t>0, x\in\mathbb{R}^n \\ u(0) = u_0(x) \end{cases}$$

$u: [0,\infty) \longrightarrow L^2(\mathbb{R}^n)$ $n = 1,2,3$; $k \in \mathbb{R}$
$V: \mathbb{R}^n \to \mathbb{R}$, $V \in L^p(\mathbb{R}^n)$ for some $p > n/2$ and $p \geq 2$.
$k: \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$, $k \in L^\infty(\mathbb{R}\times\mathbb{R}^n)$

<u>Lemma 1</u> $i\Delta : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the infinitesimal generator of a C_0 group of unitary operators.

<u>Proof</u> By Stone's Theorem, see Lecture 12 page 17, the conclusion is equivalent to $(-i)(i\Delta) = \Delta$ is self adjoint.

Recall: A symmetric operator $A: D(A) \to H$ with H a Hilbert Space is self-adjoint if and only if $\text{Range}(\pm i I_d - A) = H$

In our case for $f \in L^2(\mathbb{R}^n)$ arbitrary

$$(4) \qquad (\pm i I_d - \Delta)g = f$$

can be solved via Fourier Transform :

$$(\pm i + |\xi|^2)\,\hat{g}(\xi) = \hat{f}(\xi)$$

$$\Rightarrow \hat{g}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 \pm i}$$

$$\Rightarrow g(x) = F.T.^{-1}\left(\frac{\hat{f}(\xi)}{|\xi|^2 \pm i}\right)(x) \in H^2(\mathbb{R}^n)$$

Because $\dfrac{\hat{f}(\xi)}{|\xi|^2 \pm i} \in \{ g \in L^2(\mathbb{R}^2) : (1+|\xi|^2)\, g \in L^2(\mathbb{R}^2)\}$

So (4) has solutions for all $f \in L^2(\mathbb{R}^n) \Rightarrow$ Range $(\pm Id - \Delta) = L^2(\mathbb{R}^n) \Rightarrow \Delta : H^2 \to L^2$ is self-adjoint $\Rightarrow$ $i\Delta$ generates a C_0 group of unitary operators. QED

Lemma 2 $|u|_\Delta = \|u\|_{L^2} + \|\Delta u\|_{L^2}$ is an equivalent norm with the standard one on $H^2(\mathbb{R}^n)$.

Proof Recall:

$$\|u\|_{H^2}^2 = \|u\|_{L^2}^2 + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}^2 + \sum_{i,j=1}^{n} \left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|^2$$

But $\left\| \frac{\partial u}{\partial x_j} \right\|_{L^2}^2 = \left\langle \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_j} \right\rangle = \langle -i\xi_j \hat{u}(\xi), -i\xi_j \hat{u}(\xi) \rangle$

$= \langle \hat{u}(\xi), |\xi_j|^2 \hat{u}(\xi) \rangle$

Hence

$$\sum_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^2}^2 = \langle \hat{u}(\xi), |\xi|^2 \hat{u}(\xi) \rangle$$

Now $\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2}^2 = \langle \hat{u}(\xi), |\xi_i|^2 |\xi_j|^2 \hat{u}(\xi) \rangle$

Hence $\sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2}^2 = \langle \hat{u}(\xi), |\xi|^4 \hat{u}(\xi) \rangle$

$\Rightarrow \|u\|_{H^2}^2 = \langle \hat{u}(\xi), (1+|\xi|^2+|\xi|^4)\hat{u}(\xi) \rangle = \left\| \sqrt{1+|\xi|^2+|\xi|^4}\, \hat{u}(\xi) \right\|_{L^2}^2$

$\Rightarrow \frac{1}{2}\|\hat{u}(\xi)\|_{L^2} + \frac{1}{2}\||\xi|^2 \hat{u}(\xi)\|_{L^2} \le \|u\|_{H^2} \le \|(1+|\xi|^2)\hat{u}(\xi)\|_{L^2}$

$\Rightarrow \frac{1}{2}\|u\|_{L^2} + \frac{1}{2}\|\Delta u\|_{L^2} \le \|u\|_{H^2} \le \|u\|_{L^2} + \|\Delta u\|_{L^2}$

<u>Lemma 3</u> If $V(x) \in L^p(\mathbb{R}^n)$ with $p > n/2$ and $p \ge 2$ then for every $\varepsilon > 0$ $\exists\, C(\varepsilon) > 0$ such that

$$\|Vu\|_{L^2} \le \varepsilon \|\Delta u\|_{L^2} + C(\varepsilon)\|u\|_{L^2} \quad \forall u \in H^2(\mathbb{R}^n)$$

<u>Proof</u> By Hölder

(5) $\|Vu\|_{L^2} \le \|V\|_{L^p} \|u\|_{L^r}$ $\quad \frac{1}{r} + \frac{1}{p} = \frac{1}{2}$

Since $p > \frac{n}{2} \Rightarrow r < \frac{2n}{n-4} \Rightarrow H^2 \hookrightarrow L^r$

$$\Rightarrow \|\tilde{U}\|_{L^r} \leq C_r \|\tilde{U}\|_{H^2} \leq C_r \left(\|\tilde{U}\|_{L^2} + \|\Delta \tilde{U}\|_{L^2} \right)$$

for any $\tilde{U}$ in H^2. Let

$$\tilde{U}(x) = U(\rho x) \text{ for } \rho > 0 \quad U \in H^2$$

$$\Rightarrow \frac{1}{\rho^{n/r}} \|U(x)\|_{L^r} = \|\tilde{U}\|_{L^r} \leq C_r \left(\|\tilde{U}\|_{L^2} + \|\Delta \tilde{U}\|_{L^2} \right)$$

$$= C_r \left(\frac{1}{\rho^{n/2}} \|U\|_{L^2} + \frac{\rho^2}{\rho^{n/2}} \|\Delta U\|_{L^2} \right)$$

$$\Rightarrow \|U\|_{L^r} \leq C_r \rho^{-n(\frac{1}{2}-\frac{1}{r})} \|U\|_{L^2} + C_r \rho^{2-n(\frac{1}{2}-\frac{1}{r})} \|\Delta U\|_{L^2}$$

(5) implies $\left(\frac{1}{2} - \frac{1}{r} = \frac{1}{p} \right)$

$$\|VU\|_{L^2} \leq C_r \|V\|_{L^p} \rho^{2-n/p} \|\Delta U\|_{L^2} + C_r \|V\|_{L^p} \rho^{-n/p} \|U\|_{L^2}$$

$\forall \varepsilon > 0 \ \exists \rho > 0$ such that $\varepsilon = C_r \|V\|_{L^p} \rho^{2-n/p}$; $(2 - n/p > 0)$ and denoting $C(\varepsilon) = C_r \|V\|_{L^p} \rho^{-n/p}$ we get the conclusion.

<u>Lemma 4</u> $i(\Delta - V): H^2 \to L^2$ is the generator of a C_0 group of unitary operators.

<u>Proof</u> As in Lemma 1 (via Stone's theorem) is sufficient to prove that $\Delta - V$ is self-adjoint. Since $\Delta - V$ is symmetric (note V is real valued)

it suffices to prove Range $(\pm i I_d - \Delta + V) = L^2$ or equivalently:

$$\text{Range}\,(I_d + i(\Delta - V)) = L^2$$

and

$$\text{Range}\,(I_d - i(\Delta - V)) = L^2$$

But $i\Delta$, $-iV$ are dissipative

$\text{Range}\,(I_d - i\Delta) = L^2$ (see Lemma 1)

$\|-iVu\|_{L^2} \leq \frac{1}{2}\|i\Delta u\|_{L^2} + C\|u\|_{L^2}$ (see Lemma 3)

$\Rightarrow$ via perturbations of dissipative operators, see Lemma in Lecture 13 page 9 $\Rightarrow$

$$\text{Range}\,(I_d - i\Delta + iV) = L^2$$

Same argument for $-i\Delta$ and iV gives

$\text{Range}\,(I_d + i\Delta - iV) = L^2$ QED.

<u>Lemma 5</u> Let $f : \mathbb{R}\times H^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ $n \leq 3$,

$$f(t,u)(x) = -ik(t,x)|u(x)|^2 u(x), \quad k \in L^\infty(\mathbb{R}\times\mathbb{R}^n).$$

Then f is well defined and $\exists\, C > 0$ such that

$$\|f(u) - f(v)\|_{H^2} \leq C\left(\|u\|^2_{H^2} + \|v\|^2_{H^2}\right)\|u - v\|_{H^2}$$

Proof

$$f(u) - f(v) = -ik\left(|u|^2 u - |v|^2 v\right)$$
$$= -ik\left[(\bar{u}u + \bar{u}v)(u-v) + v^2(\bar{u} - \bar{v})\right]$$

So

$$\|f(u) - f(v)\|_{L^2} \leq |k|\Big(\|\bar{u}u + \bar{u}v\|_{L^\infty} \|u-v\|_{L^2}$$
$$+ \|v^2\|_{L^\infty} \|u-v\|_{L^2}\Big)$$
$$\leq 2|k|\left(\|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2\right)\|u-v\|_{L^2}$$

Since $H^2(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $n \leq 3$ we obtain.

(*) $$\|f(u) - f(v)\| \leq 2|k|\, C\left(\|u\|_{H^2}^2 + \|v\|_{H^2}^2\right)\|u-v\|_{L^2}$$

Similarly we have:

$$\|\Delta(f(u) - f(v))\|_{L^2} \leq |k|\left(2\|u\|_{L^\infty}^2 + 2\|v\|_{L^\infty}^2\right)\|\Delta(u-v)\|_{L^2}$$
$$+ 2|k|\left(\|\nabla(\bar{u}u + \bar{u}v)\|_{L^4} + \|\nabla v^2\|_{L^4}\right)\|\nabla(u-v)\|_{L^4}$$
$$+ |k|\left(\|\Delta(\bar{u}u + \bar{u}v)\|_{L^2} + \|\Delta v^2\|_{L^2}\right)\|u-v\|_{L^\infty}$$

which by $H^2(\mathbb{R}^n) \hookrightarrow W^{1,4}(\mathbb{R}^n)$ for $n \leq 3$ implies

$$\| \Delta(f(U)-f(V))\|_{L^2} \leq \tilde{C}\left(\|U\|_{H^2}^2 + \|V\|_{H^2}^2\right)\|U-V\|_{H^2}$$

Adding this to $(*)$

$$\Rightarrow \|f(U)-f(V)\|_{H^2} \leq |f(U)-f(V)|_{\Delta} \leq C_1\left(\|U\|_{H^2}^2+\|V\|_{H^2}^2\right) \cdot \|U-V\|_{H^2} \quad \text{QED}$$

<u>Theorem</u> If $n \leq 3$, V is real valued $V \in L^p(\mathbb{R}^n)$, for some $p > n/2$ and $p \geq 2$, $k \in L^\infty(\mathbb{R}\times\mathbb{R}^n)$ and $U_0 \in H^2(\mathbb{R}^n)$ then (3) has a unique classical maximal solution:

$$U : (T_{min}, T_{max}) \longrightarrow \mathbb{C}$$

such that

$$T_{min} < 0 < T_{max}$$

$$T_{min} = -\infty \text{ or } \lim_{t \searrow T_{min}} \|U(t)\|_{H^2} = +\infty$$

$$T_{max} = +\infty \text{ or } \lim_{t \nearrow T_{max}} \|U(t)\|_{H^2} = +\infty$$

<u>Proof</u> Lemmas 2, 4 and 5 imply the hypotheses of Theorem 4 at page 5 which implies the conclusion.