Lecture 2 Math 555

Note Title 1/16/2008

1. Bifurcations in nonlinear PDE's

1.1. Contraction Principle in metric spaces
1.2. Calculus in Banach spaces and the implicit function theorem
1.3. An application to Schrödinger Eq.
1.4. Short description of the first project

1.1. Contraction principle in metric spaces

Def. (Metric space) A set M together with a function $d: M \times M \to \mathbb{R}$ such that:

1. $d(x,y) \geq 0 \quad \forall x, y \in M$, $d(x,y) = 0$ if and only if $x = y$
2. $d(x,y) = d(y,x) \quad \forall x, y \in M$
3. $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \in M$

is called a metric space

Remark Any metric space can be organized as a topological space by defining the open sets via:

$$U \text{ is open} \Leftrightarrow \forall x \in U \ \exists \varepsilon_x > 0 \text{ such that } \{y \in M \mid d(x,y) < \varepsilon_x\} \subseteq U$$

Def (convergence in metric spaces) $\{x_n\}_{n \in \mathbb{N}} \subseteq M$ is convergent if $\exists x_0 \in M$ such that:

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N} \text{ such that } d(x_n, x_0) < \varepsilon \quad \forall n \geq N_\varepsilon$$

Notation: $\lim_{n \to \infty} x_n = x_0$ and x_0 is unique.

Def (fundamental sequences) $\{x_n\}_{n \in \mathbb{N}} \subseteq M$ is fundamental (Cauchy) if

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N} \text{ such that } d(x_n, x_m) < \varepsilon \quad \forall n, m \geq N_\varepsilon$$

Def (complete metric spaces) A metric space is complete if and only if every fundamental sequence in it is convergent.

__Def__ (contraction) A map $F: M \to M$ where (M, d) is a metric space is called a (strict) contraction if $\exists\ 0 \leq L < 1$ such that

$$d(F(x), F(y)) \leq L\, d(x,y) \quad \forall x, y \in M$$

__Remark__ Any contraction is a continuous map.

__Theorem (contraction Principle)__ Any contraction on a complete metric space has a unique fixed point. In other words, if $F: M \to M$ is a contraction and (M, d) is a complete metric space then the eq:

$$F(x) = x$$

has a unique solution.

__Proof__ Existence of the fixed point:

Fix $x_1 \in M$ consider the sequence defined inductively

$$x_{n+1} = F(x_n) \quad n \geq 1$$

Note: $d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1}))$
$\leq L\, d(x_n, x_{n-1}) \leq \dots$
$\leq L^{n-1} d(x_2, x_1)$

Claim $\{x_n\}_{n\in\mathbb{N}}$ is fundamental.

Indeed for $1 \leq n < m$

$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m) \leq \dots$
$\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m)$
$\leq L^{n-1} d(x_2, x_1) + L^n d(x_2, x_1) + \dots + L^{m-2} d(x_2, x_1)$
$= L^{n-1} (1 + L + \dots + L^{m-n-1})\, d(x_2, x_1)$
$\leq L^{n-1} \cdot \frac{1}{1-L}\, d(x_2, x_1)$ because $L<1$

Since $L^n \longrightarrow 0$ as $n \to \infty$ $(0 \leq L < 1)$, for any $\varepsilon > 0$ we can choose $N_\varepsilon \in \mathbb{N}$ such that

$$\frac{L^{n-1}}{1-L}\, d(x_2, x_1) < \varepsilon \quad \forall\, n \geqslant N_\varepsilon$$

hence

$d(x_n, x_m) \leq \varepsilon \quad \forall\, n, m \geqslant N_\varepsilon$ and $\{x_n\}_{n\in\mathbb{N}}$ is fundamental

Since (M, d) is complete $\exists x_0 \in M$ such that $\lim_{n \to \infty} x_n = x_0$.

Pass to the limit as $n \to \infty$ in

$$\begin{array}{ccl} x_{n+1} & = & F(x_n) \\ \downarrow & & \downarrow \text{ by continuity of } F \\ x_0 & = & F(x_0) \\ & \uparrow & \\ & \text{by uniqueness of the limit} & \end{array}$$

Hence x_0 is a fixed point of F.

<u>Uniqueness of the fixed point</u>: assume there are two solutions:

$$F(x_0) = x_0$$
$$F(x_1) = x_1$$

Then $d(x_0, x_1) = d(F(x_0), F(x_1)) \underset{\text{by contraction}}{\leq} L\, d(x_0, x_1)$
$\Rightarrow (1-L)\, d(x_0, x_1) \leq 0 \Rightarrow d(x_0, x_1) = 0 \Rightarrow x_0 = x_1$

The theorem is completely proven QED

1.2. Calculus in Banach spaces

Let X, Y be two Banach spaces over real or complex numbers and $B(X, Y)$ the space of linear, bounded operators from X into Y.

Let $F : U \longrightarrow Y$, $U \subseteq X$ open be a map.

<u>Def</u> F has Gâteaux derivative at $u \in U$ in the direction $x \in X$ if and only if:

$$\lim_{\varepsilon \to 0} \frac{F(u+\varepsilon x) - F(u)}{\varepsilon} \text{ exists and is finite}$$

Notation $$dF(u)[x] = \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon x) - F(u)}{\varepsilon}$$

<u>Def</u> F is Gâteaux differentiable at $u \in U$ if and only if it has Gâteaux derivative in any direction.

Def F is Fréchet differentiable at $v \in U$ if there exists $A \in \mathcal{B}(X; Y)$ such that:

$$\lim_{\|x\| \to 0} \frac{\|F(v+x) - F(v) - Ax\|}{\|x\|} = 0$$

Notation: $A = DF(v)$ is called the Fréchet derivative

Theorem If F is Fréchet differentiable at $v \in U$ (with Fréchet derivative A) then F is Gâteaux differentiable at v and

$$Ax = dF(v)[x] \quad \forall x \in X$$

Proof We have:

$$\frac{F(v+\varepsilon x) - F(v)}{\varepsilon} = \operatorname{sgn}(\varepsilon)\|x\| \frac{F(v+\varepsilon x) - F(v) - A\varepsilon x}{\|\varepsilon x\|} + \frac{A\varepsilon x}{\varepsilon}$$

Passing to the limit when $\varepsilon \to 0$ we get

$$\lim_{\varepsilon \to 0} \frac{F(v+\varepsilon x) - F(v)}{\varepsilon} = Ax \qquad \text{QED.}$$

<u>Remark</u> Gâteaux differentiable $\not\Rightarrow$ Fréchet differentiable. Example:

$$f:\mathbb{R}^2\to\mathbb{R} \quad f(x,y)=\begin{cases}\frac{x^2y}{x^4+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

$$df(0,0)[(a,b)]=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{(\varepsilon a)^2(\varepsilon b)}{(\varepsilon a)^4+(\varepsilon b)^2}=\begin{cases}\frac{a^2}{b} & \text{if } b\neq 0\\ 0 & \text{if } b=0\end{cases}$$

$df(0,0):\mathbb{R}^2\to\mathbb{R}$ is not linear so f cannot be Fréchet differentiable.

<u>Theorem</u> If F is Fréchet differentiable at $v\in U$ then F is continuous at v.

<u>Proof</u> Let $\{x_n\}_{n\in\mathbb{N}}\subset U$, $\lim_{n\to\infty} x_n=v$. Then

$$\begin{aligned}F(x_n)-F(v) &= F(v+x_n-v)-F(v)=\\ &= F(v+x_n-v)-F(v)-A(x_n-v)+A(x_n-v)\\ &= \|x_n-v\|\cdot\frac{F(v+x_n-v)-F(v)-A(x_n-v)}{\|x_n-v\|}+\\ &\quad + A(x_n-v)\end{aligned}$$

Since A is bounded $A(x_n-v)\to 0$ as $n\to\infty$ hence

$\lim_{n \to \infty} (F(x_n) - F(v)) = 0$. Since $x_n \to v$ was arbitrary $\Rightarrow F$ is continuous at $v \in U$. QED

<u>Remark</u> For $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

we have $df(0,0)[(a,b)] = 0 \quad \forall (a,b) \in \mathbb{R}^2$
hence the Gâteaux derivative is linear and bounded but f is still not Fréchet differentiable at $(0,0)$ because it is not continuous: $\lim_{n \to \infty} f(\frac{1}{n}, \frac{1}{n^3}) = \frac{1}{2} \neq (0,0)$

<u>Theorem</u> Let $V \subseteq U$ be a neighborhood of $v \in U$. Assume $dF(v)[\cdot] \in B(X,Y)$ $\forall v \in V$ and $dF: V \to B(X,Y)$ is continuous. Then F is Fréchet differentiable at v.

<u>Proof</u> We first need an extension of the mean value theorem:

__Lemma__ If $u, v \in X$ and F has Gâteaux derivative in direction $v-u$ for all the points in the segment:

$$[u,v] = \{x \in X \mid x = (1-t)u + tv \text{ for some } 0 \leq t \leq 1\}$$

then $\exists\, x \in [u,v] \setminus \{u,v\}$ such that

$$\| F(v) - F(u) \| \leq \| dF(x)[v-u] \|$$

__Proof of Lemma__ Let $y^* \in Y^*$ such that

$$\|y^*\| = 1 \text{ and } y^*(F(v) - F(u)) = \|F(u) - F(v)\|.$$

(Such a functional exists via Hahn-Banach Theorem)

Let $f : [0,1] \to \mathbb{R}$ (or $\mathbb{C}$) $f(t) = y^* F((1-t)u + tv)$.
Now, via direct calculation, f is differentiable on $[0,1]$:

$$f'(t) = y^* dF((1-t)u + tv)[v-u].$$

hence, via mean value theorem for f, $\exists\, t \in (0,1)$:

$$f(1) - f(0) = f'(t) \quad \text{(for } f \text{ real valued)}$$
$$|f(1) - f(0)| \leq |f'(t)| \quad \text{(for } f \text{ complex valued)}$$

In both cases we have

$$|y^*F(v) - y^*F(u)| \leq |y^* dF((1-t)u+tv)[v-u]|$$

$$\Rightarrow \|F(v)-F(u)\| \leq \|y^*\| \, \|dF((1-t)u+tv)[v-u]\|$$
$$\leq \|dF((1-t)u+tv)[v-u]\|$$

The Lemma is completely proven.

Back to the theorem: $\exists \varepsilon > 0$ such that:
$$B(u,\varepsilon) = \{x \in X \mid \|x-u\| < \varepsilon\} \subseteq V$$
Consider
$$G : B(u,\varepsilon) \longrightarrow Y$$
$$G(v) = F(v) - F(u) - dF(u)[v-u].$$
A direct calculation shows that G is Gâteaux differentiable on $B(u,\varepsilon)$ with:
$$dG(v)[x] = dF(v)[x] - dF(u)[x]$$
(here we used the linearity of $dF(u)$).
Apply previous Lemma on segments $[u,v] \subset B(u,\varepsilon)$
$\Rightarrow \forall v \in B(u,\varepsilon) \ \exists t_v \in (0,1)$ such that:
$$\|G(v) - G(u)\| \leq \|dG((1-t_v)u + t_v v)[v-u]\|$$
$$\Leftrightarrow \quad \|F(v) - F(u) - dF(u)[v-u]\| \leq$$
$$\leq \|\{dF((1-t_v)u + t_v v) - dF(u)\}[v-u]\|$$
$$\leq \|dF((1-t_v)u + t_v v) - dF(u)\| \, \|v-u\|$$
hence for $v \neq u$:
$$\frac{\|F(v) - F(u) - dF(u)[v-u]\|}{\|v-u\|} \leq \|dF((1-t_v)u + t_v v) - dF(u)\|$$

For $v \to u$ we have $(1-t_v)u + t_v v \to u$ and the RHS above converges to zero due to the continuity of $dF: B(u,\varepsilon) \to B(X,Y)$. Hence the LHS in the relation above converges to zero as $v \to u$ which is equivalent with the existence of Fréchet derivative at u. QED.

<u>Remark</u> Both Gâteaux and Fréchet differentiable functions verify:

(i) $d(F+G)(u) = dF(u) + dG(u)$

$D(F+G)(u) = DF(u) + DG(u)$

(ii) $d(\lambda F)(u) = \lambda\, dF(u) \quad \forall \lambda \in \mathbb{R}\ (\text{or } \mathbb{C})$

$D(\lambda F)(u) = \lambda\, DF(u) \quad \forall \lambda \in \mathbb{R}\ (\text{or } \mathbb{C})$

In addition the Fréchet differential satisfies

(iii) $D(F \circ G)(u) = DF(G(u)) \circ DG(u)$

but this property may fail for Gâteaux differentials.

$d(F \circ G)(u)$ might not exist even when $dF(G(u))$ and $dG(u)$ exists !!!

<u>Remark</u> Integral calculus can be also recovered. For example if $dF(\cdot)[v-u]$ is continuous on the segment $[u, v]$ then

$$F(v) - F(u) = \int_0^1 dF((1-t)u + tv)[v-u]\, dt$$