Summary

1.2 The implicit function theorem in Banach spaces and Lyapunov-Schmidt reduction

Def Let $F: X \times Y \to Z$ X, Y, Z Banach spaces and $(x_0, y_0) \in X \times Y$. F is Fréchet diff at (x_0, y_0) w.r.t variable x if and only if

$$g: X \to Z \quad g(x) = F(x, y_0)$$

is Fréchet differentiable at x_0.

Notation $DF_x(x_0, y_0) = Dg(x_0)$

Theorem Let $f: U \to Z$ where $U \subseteq X \times Y$ is open and X, Y, Z are Banach spaces. Assume:

1° $f(x_0, y_0) = 0$, $(x_0, y_0) \in U$

2° f is continuous at (x_0, y_0).

3° $Df_x(x_0, y_0)$ exists and is an isomorphism (from X to Z)

4° $Df_x(x, y)$ exists for all $(x, y) \in U$ and $(x, y) \longmapsto Df_x(x, y)$ is continuous at (x_0, y_0) as a map from U to $B(X, Z)$ = Banach space of linear bounded operators from X to Z.

Then:

(i) $\exists \delta > 0, r > 0$ with $\overline{B_\delta(x_0)} \times \overline{B_r(y_0)} \subseteq U$ and a map $U : \overline{B_r(y_0)} \longrightarrow \overline{B_r(x_0)}$ such that $(U(y), y)$ are the only solutions of:

$$f(x,y) = 0 \quad \text{in } \overline{B_\delta(x_0) \times B_r(y_0)}$$

If in addition f is continuous on $\overline{B_\delta(x_0)} \times \overline{B_r(y_0)}$ then U is continuous.

(ii) If f is C^1 on $B_\delta(x_0) \times B_r(y_0)$, i.e. $Df(x,y)$ exist for all $(x,y) \in B_\delta(x_0) \times B_r(y_0)$ and $(x,y) \longmapsto Df(x,y)$ is continuous from $B_\delta(x_0) \times B_r(y_0)$ to $B(X \times Y, Z)$, then U is C^1 and:

$$DU(y) = -\left[Df_x(U(y), y)\right]^{-1} \circ Df_y(U(y), y)$$

(iii) If f is C^p, $p \geq 2$, on $B_\delta(x_0) \times B_r(y_0)$ then U is C^p

Sketch of proof: Denote $A = Df_x(x_0, y_0)$ and

$$g : U \longrightarrow X \quad g(x,y) = x - A^{-1} f(x,y)$$

Then $D_x g$ exists:

$$D_x g = I_d - A^{-1} D_x f(x,y)$$

where $Id : X \to X$ is $Id(x) = x$. In particular
$$D_x g(x_0, y_0) = 0$$

Since $D_x f$ is continuous at (x_0, y_0) as a map from U to $B(X, Z)$ we get that $D_x g$ is continuous at (x_0, y_0) as a map from U to $B(X, X)$. Hence we can choose $\delta > 0$ and $r_1 > 0$ such that
$$\| D_x g(x, y) \| \le \frac{1}{2} \quad \forall\, x, y,\ \|x - x_0\| \le \delta,\ \|y - y_0\| \le r_1$$

Furthermore: $\left.\begin{array}{l} g(x_0, y_0) = x_0 \\ g \text{ continuous at } (x_0, y_0) \end{array}\right\} \Rightarrow$
we can choose $0 < r \le r_1$ such that
$$\| g(x_0, y) - x_0 \| \le \frac{1}{2}\delta \quad \forall\, y,\ \|y - y_0\| \le r$$

<u>Claim</u> $\forall\, y \in \overline{B_r(y_0)}$, $g_y : \overline{B_\delta(x_0)} \to \overline{B_\delta(x_0)}$ given by
$$g_y(x) = g(x, y)$$

is well defined and a contraction.
<u>Proof of Claim</u>: Clearly $g_y : \overline{B_\delta(x_0)} \to X$

By applying the mean value theorem, see

Lemma in Lecture 2 we have for $x_1, x_2 \in B_\delta(x_0)$

$$\|g_y(x_1) - g_y(x_2)\| = \|g(x_1, y) - g(x_2, y)\|$$

$\leq \|Dg_x(x, y)\| \, \|x_1 - x_2\|$ for some $x \in [x_1, x_2] \subseteq \overline{B_\delta(x_0)}$. Also $y \in \overline{B_r(y_0)}$ so $\|Dg_x(x,y)\| \leq \frac{1}{2}$ and, consequently

$$\|g_y(x_1) - g_y(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|$$

i.e. g_y is a contraction. It remains to show that the range of $g_y \subseteq \overline{B_\delta(x_0)}$. Let $x \in \overline{B_\delta(x_0)}$

$$\|g_y(x) - x_0\| = \|g(x,y) - g(x_0, y_0)\| =$$
$$= \|g(x,y) - g(x_0,y) + g(x_0,y) - g(x_0,y_0)\|$$
$$\leq \|g_y(x) - g_y(x_0)\| + \|g(x_0,y) - g(x_0,y_0)\|$$
$$\leq \frac{1}{2}\|x - x_0\| + \frac{1}{2}\delta \leq \delta$$

The claim is completely proven.

By contraction principle, for any $y \in \overline{B_r(y_0)}$ g_y has a unique fixed point in $\overline{B_\delta(x_0)}$, call it $v(y)$. Hence the equation:

$$x = g(x,y) = x - A^{-1} f(x,y)$$

or, equivalently $Ax = \underline{Ax - f(x,y)} \Leftrightarrow f(x,y)=0$ has a unique sln in $\overline{B_\delta(x_0)}$, $x = v(y)$ for any fixed $y \in \overline{B_r(y_0)}$.

Part (i) is proven except continuity of $y \longmapsto v(y)$. The continuity follows immediately from the general result:

<u>Lemma</u>: Let $F_\lambda : K \to K$ be a family of uniform contractions (all have the same Lipschitz constant) where K is a closed set in the complete metric space M and $\lambda \in \Lambda$ is a topological space. Assume that F_λ depends continuously on λ, i.e. $\forall x \in K, \forall \lambda_0 \in \Lambda, \forall \varepsilon > 0$ there exists $W(\varepsilon, x, \lambda_0)$ a neighborhood of λ_0 such that

$$\|F_\lambda(x) - F_{\lambda_0}(x)\| < \varepsilon \quad \forall \lambda \in W(\varepsilon, x, \lambda_0).$$

Then the unique fixed point of F_λ depends continuously on λ.

<u>HW</u>: Prove the lemma and use it to show $y \longmapsto v(y)$ is continuous.

Part (ii) We first show that $D_x f(x,y)$ is an isomorphism (from $X \to Z$) for all $(x,y) \in B_\delta(x_0) \times B_r(y_0)$. It suffices to show that $A^{-1} D_x f$ is an isomorphism because $D_x f = A(A^{-1} D_x f)$ will then be a composition of isomorphisms.

We have already shown

$$A^{-1} D_x f = I_d - D_x g$$

and $\| D_x g \| \leq \frac{1}{2}$ on $B_\delta(x_0) \times B_r(y_0)$

The result now follows from the general one:
If $L : X \to X$ is a bounded linear operator on the Banach space X with $\|L\| < 1$ then $I_d - L$ is an isomorphism on X with inverse:

$$(I_d - L)^{-1} = I_d + L + L^2 + \dots + L^n + \dots$$

Hw Prove this result using contraction principle.

Once $D_x f(x, y)$ is an isomorphism for all (x, y) in $B_\delta(x_0) \times B_r(y_0)$ one can directly show that

$$-\left[D_x f(v(y), y) \right]^{-1} \circ D_y f(v(y), y)$$

is the Fréchet derivative of v at y by using the definition of the Fréchet derivative for f and the eq:

$$f(v(y), y) = 0$$

See for example L. Nirenberg: Topics in Nonlinear Functional Analysis, Section 2.7.

Part (iii) Follows directly by differentiating the formula for Dv proved in part (ii). QED.

Lyapunov - Schmidt reduction

Consider $f: U \to Y$, $U \subseteq X \times \Lambda$ open
X, Λ, Y Banach spaces and assume
$$f(x_0, \lambda_0) = 0 \quad , (x_0, \lambda_0) \in U$$
Study the solutions of

(1) $$f(x, \lambda) = 0 \quad \text{in } U$$

Assume f is C^1 on U and $D_x f(x_0, y_0)$ is Fredholm:
(a) $\ker D_x f(x_0, y_0) = X_1$ is finite dimensional

(b) range $D_x f(x_0, y_0) = Y_1$ is a closed subspace of Y with finite codimension.

The procedure reduces the infinite dimensional equation (1) to a finite dimensional equation:

Consider the decompositions
$$X = X_1 \oplus X_2 \ , \ Y = Y_1 \oplus Y_2 .$$
with associated continuous projections

$$P: X \to X_2 \text{ and } Q: Y \to Y_1$$

(1) is equivalent to :

$$\begin{cases} Q f(x_1 + x_2, \lambda) = 0 & (2.1) \\ (I_d - Q) f(x_1 + x_2, \lambda) = 0 & (2.2) \end{cases}$$

Let $x_0 = x_1^0 + x_2^0$, $x_1 \in X_1$, $x_2 \in X_2$ then $\exists\, r_1, r_2, r_\lambda$ such that $(x_1 + x_2, \lambda) \in U$ whenever $(x_2, x_1, \lambda) \in B_{r_2}(x_2^0) \times B_{r_1}(x_1^0) \times B_{r_\lambda}(\lambda_0) \subseteq X_2 \times X_1 \times \Lambda$

Let $F : B_{r_2}(x_2^0) \times B_{r_1}(x_1^0) \times B_{r_\lambda}(\lambda_0) \to Y_1$ given by

$$F(x_2, x_1, \lambda) = Q f(x_1 + x_2, \lambda)$$

Then

$$F(x_2^0, x_1^0, \lambda_0) = Q f(x_0, \lambda_0) = Q\, 0 = 0$$

$$D_{x_2} F(x_2^0, x_1^0, \lambda_0) = Q\, D_x f(x_0, \lambda_0)\, i_{x_1^0}$$

where $i_{x_1^0} : X_2 \to X \quad i_{x_1^0}(x) = x_1^0 + x$

We have

$$\ker D_{x_2} F(x_2^0, x_1^0, \lambda_0) = \ker Q\big|_{Y_1} \cup \ker D_x f(x_0, y_0)\big|_{X_2} = \{0\}$$

$$\text{range } D_{x_2} F(x_2^0, x_1^0, \lambda_0) = Y_1$$

$D_{x_2} F(x_2^0, x_1^0, \lambda_0) : X_2 \to Y$ continuous because $i_{x_1^0}$, Q and $D_x f$ are

By the open mapping theorem we have that $D_{x_2}F(x_2^0, x_1^0, \lambda_0)$ is an isomorphism from X_2 to Y_1

F and $D_{x_2}F$ are continuous on $B_{r_2}(x_2^0) \times B_{r_1}(x_1^0) \times B_{r_\lambda}(\lambda_0)$

Implicit function theorem applies to eq 2-1 and there exists $\tilde{r}_2, \tilde{r}_1, \tilde{r}_\lambda > 0$ and a C^1 map

$$x_2 : B_{\tilde{r}_1}(x_1^0) \times B_{\tilde{r}_\lambda}(\lambda_0) \longmapsto B_{\tilde{r}_2}(x_2^0)$$

such that

$(x_2(x_1, \lambda), x_1, \lambda)$ are the only sol's of (2.1) in $B_{\tilde{r}_2} \times B_{\tilde{r}_1} \times B_{\tilde{r}_\lambda}$ neighborhood of (x_0, λ_0)

Plugging in (2.2) we get:

$$(I_d - Q) f(x_1 + x_2(x_1, \lambda), \lambda) = 0$$

The left hand side is C^1 in x_1, λ and defined on a subset of $X_1 \times \Lambda$ with values in Y_2. X_1 and Y_2 are both <u>finite dimensional</u>.

<u>Note that</u>: For $G(x_1, \lambda) = (I_d - Q) f(x_1 + x_2(x_1, \lambda), \lambda)$ we have:

$$G(x_1^0, \lambda_0) = (I_d - Q) f(x_0, \lambda_0) = 0$$

and

$$D_{x_1} G(x_1^0, \lambda_0) = (I_d - Q)\, D_x f(x_0, \lambda_0) \circ [I_d + D_{x_1} x_2]$$
$$= 0$$

Because range of $D_x f(x_0, \lambda_0) = Y_1 = \ker(I_d - Q)$

So if one wants to use again implicit function theorem for (2.2) then one should look for $\lambda = \lambda(x_1)$:

$$D_\lambda G(x_1^0, \lambda_0) = \cancelto{0}{(I_d - Q)\, D_x f(x_0, \lambda_0) \circ D_\lambda x_2}$$
$$+ (I_d - Q)\, D_\lambda f(x_0, \lambda_0).$$

<u>Conclusion</u> : The Lyapunov-Schmidt reduction makes a one to one correspondence between the solutions of $f(x, \lambda) = 0$ $x \in X, \lambda \in \Lambda$
and the solutions of $\dim Y_1$ equations in $\dim X_1 + \dim \Lambda$ variables in a small neighborhood of one sln (x_0, λ_0) provided f is locally C^1 and $D_x f(x_0, \lambda_0)$ is <u>Fredholm</u>.