1.4. An application to Schrödinger Eq

(1) $i\dfrac{\partial u}{\partial t} = -\Delta u + Vu + \gamma |u|^2 u$

$u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C} \qquad u(t,\cdot) \in L^2(\mathbb{R}^n)$

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$$

$V : \mathbb{R}^n \to \mathbb{R} \qquad |V(x)| \le \dfrac{C}{(1+|x|)^{3+0}}$

$\gamma \in \mathbb{R}$.

Look for bound states:

$u(t,x) = e^{-i\lambda t}\varphi(x) \qquad \varphi \in H^2(\mathbb{R}^n), \lambda \in \mathbb{R}$

Plug in in (1):

(2) $\lambda \varphi = -\Delta \varphi + V\varphi + \gamma |\varphi|^2 \varphi$

(2) is equivalent to

(3) $F(\varphi,\lambda) = (-\Delta + V - \lambda)\varphi + \gamma|\varphi|^2\varphi = 0$

$F : H^2 \times \mathbb{R} \longrightarrow L^2$

— F is well defined:

$V\varphi \in L^2$ because $V \in L^\infty$ and $\varphi \in L^2$

$|\varphi|^2\varphi \in L^2 \Leftrightarrow \varphi \in L^6$ true because

$H^2(\mathbb{R}^n) \hookrightarrow L^6(\mathbb{R}^n)$

for $n = 1,2,3,4,5$

(Sobolev Imbedding $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q$ for $p \le q \le \frac{np}{n-mp}$ if $mp < n$)

— Gateaux derivative in φ:

$$dF_\varphi(\varphi_0,\lambda_0)\psi = \lim_{\varepsilon\to 0} \frac{F(\varphi_0+\varepsilon\psi,\lambda_0) - F(\varphi_0,\lambda_0)}{\varepsilon}$$

$$= (-\Delta + V - \lambda_0)\psi + 2\gamma|\varphi_0|^2\psi + \lim_{\varepsilon\to 0}\frac{\gamma\varphi_0^2\bar{\psi}\bar{\varepsilon}}{\varepsilon}$$

where $\bar{\varepsilon}$ = complex conjugate of $\varepsilon \in \mathbb{C}$

The limit does not exist for $\varepsilon \in \mathbb{C}$ but exists for $\varepsilon \in \mathbb{R}$. We need to consider

$H^2(\mathbb{R}^n, \mathbb{C})$ and $L^2(\mathbb{R}^n, \mathbb{C})$ over the field $\mathbb{R}$! In this case:

$$dF_\varphi(\varphi_0, \lambda_0)\Psi = (-\Delta + V - \lambda_0)\Psi + 2\gamma|\varphi_0|^2\Psi + \gamma\varphi_0^2\overline{\Psi}$$

and $dF_\varphi(\varphi_0, \lambda_0) : H^2 \to L^2$ is linear, bounded and depends continuously on φ_0 and λ_0.

$\Rightarrow$ Fréchet derivative exists:

$$DF_\varphi(\varphi_0, \lambda_0)\,\cdot = (-\Delta + V - \lambda_0 + 2\gamma|\varphi_0|^2 + \gamma\varphi_0^2\,\overline{})\,\cdot$$

and depends continuously in (φ_0, λ_0).

<u>Remark</u> Since $DF_\lambda(\varphi_0, \lambda_0)\delta\lambda = \varphi_0\,\delta\lambda$ exists and is continuous in (φ_0, λ_0) we deduce that F is C^1.

We also have:

$$F(0,\lambda)=0 \quad \forall \lambda\in\mathbb{R}$$

$$DF_{\varphi}(0,\lambda)=(-\Delta+V-\lambda): H^2\to L^2$$

Implicit function theorem applies at $\lambda\in\mathbb{R}$ for which $(-\Delta+V-\lambda)$ is an isomorphism

— Spectral properties of $-\Delta+V$

1° $-\Delta+V$ is s.a. on L^2 with domain H^2 hence the spectrum is a subset of the real line

2° The spectrum is bounded from below

3° $[0,\infty)$ is in the (essential) spectrum, although $-\Delta+V-\lambda$, $\lambda>0$ is one to one.

4° The spectrum on $(-\infty,0)$ could be empty or made up only of e-values with finite multiplicity, λ. In this case

$$\text{Range}(-\Delta+V-\lambda)=\text{Ker}(-\Delta+V-\lambda)^{\perp}$$

A particular case in which $-\Delta+V$ has negative e-values is V negative in dimensions 1 and 2. In three dimensions or higher one needs V sufficiently large and negative (see Reed & Simon Vol 4)

5° If $-\Delta + V$ has negative e-values then the lowest one is simple and its e-vector can be chosen strictly positive.

<u>Remark</u> $(-\Delta + V - \lambda) : H^2 \to L^2$ is an isomorphism for $\lambda < 0$, λ not an e-value. Hence by the implicit function theorem, there are no small amplitude ($\|\psi\|_{H^2}$ suff small but nonzero) solutions of (2) near $\lambda < 0$, λ not an e-value.

We focus on the bifurcation from the lowest e-values, $\lambda_0 < 0$, or more generally any simple negative e-value assuming it exist.

<u>Hypothesis</u> $\lambda_0 < 0$ is the lowest e-value of $-\Delta + V$ with the corresponding e-vector $\psi_0 \in H^2$, $\psi_0 > 0$.

<u>Remark</u> If λ_0 is not the lowest e-value but it is simple we can still choose ψ_0 real valued via:

$$(-\Delta + V - \lambda_0)\psi_0 = 0 \Rightarrow (-\Delta + V - \lambda_0)\overline{\psi_0} = 0$$

$\Rightarrow$ $\psi_0 + \overline{\psi_0}$ is real valued and an e-vector.

— Lyapunov - Schmidt Reduction

$$H^2 = \mathrm{Span}_{\mathbb{R}}\{\psi_0, i\psi_0\} \oplus (\psi_0^{\perp} \cap H^2)$$

(the orthogonal complement is with respect to the complex scalar product)

$$L^2 = \psi_0^{\perp} \oplus \mathrm{Span}_{\mathbb{R}}\{\psi_0, i\psi_0\}.$$

Let $P_{\perp} : L^2 \to \psi_0^{\perp}$ be the orthogonal projection

(3) is equivalent to:

$$\begin{cases} P_{\perp} F(a_1\psi_0 + ia_2\psi_0 + h, \lambda) = 0 & (3.1) \\ \langle \psi_0, F(a_1\psi_0 + ia_2\psi_0 + h, \lambda)\rangle = 0 & (3.2) \end{cases}$$

By the standard Lyapunov - Schmidt reduction (3.1) gives:

$$h = h(a_1, a_2, \lambda), \quad h : \underbrace{B_\delta(0)}_{\subseteq \mathbb{R}\times\mathbb{R}} \times \underbrace{B_{\delta_1}(\lambda_0)}_{\subseteq \mathbb{R}} \to \psi_0^{\perp} \cap H^2$$

For simplicity we identify $(a_1, a_2) = a_1 + ia_2$

In order to solve (3.2) we need some properties of h:

$$h(0,\lambda) = 0 \quad \begin{array}{l}(\text{since } h \text{ is unique and}\\ F(0,\lambda) \equiv 0).\end{array}$$

$$h(e^{i\theta}a, \lambda) = e^{i\theta} h(a,\lambda)$$
$$h(\bar{a}, \lambda) = \overline{h(a,\lambda)}$$

hence $h(a,\lambda)$ is real valued for real valued a, $|a|<\delta$.

This is because (3.1) inherits the symmetries of (3) (called O_2 or gauge symmetries):

$$F(e^{i\theta}\psi, \lambda) = e^{i\theta} F(\psi,\lambda).$$
$$F(\bar{\psi}, \lambda) = \overline{F(\psi,\lambda)}$$

So if $(a\psi_0 + h(a,\lambda), \lambda)$ is a slu of (3.1) then

$$P_\perp F(e^{i\theta} a\psi_0 + e^{i\theta} h(a,\lambda), \lambda)$$
$$= P_\perp e^{i\theta} F(a\psi_0 + h(a,\lambda),\lambda) = 0$$

i.e.

$$(e^{i\theta} a\psi_0 + e^{i\theta} h(a,\lambda), \lambda) \text{ is a slu.}$$

By uniqueness $h(e^{i\theta}a, \lambda) = e^{i\theta} h(a,\lambda)$.
A similar argument shows $h(\bar{a},\lambda) = \overline{h(a,\lambda)}$.

(3.1) can be written:

$$(-\Delta + V - \lambda)h + P_\perp\left[|a\psi_0 + h|^2 (a\psi_0 + h)\right] = 0$$

or $h = (-\Delta + V - \lambda)^{-1} P_\perp \left[|a\psi_0 + h|^2 (a\psi_0 + h)\right]$

Now, because for $a \in \mathbb{R}$, $|a| < \delta$:

$$h(0, \lambda) = 0$$

and $\frac{\partial h}{\partial a}(0,\lambda) = (-\Delta + V - \lambda)^{-1} P_\perp\, 3\psi_0 h^2 \Big|_{a=0} = 0$

$\Rightarrow h(a,\lambda) = o(|a|)$ and plugged in above we get: $h(a,\lambda) = \mathcal{O}(|a|^3)$.

Then for $a \in \mathbb{R}$, $|a| < \delta$:

$$\frac{\partial h}{\partial \lambda} = \left[-\Delta + V - \lambda + \gamma(3a^2\psi_0^2 + 6a\psi_0 h + 3h^2)\right]^{-1} h$$

$$= \mathcal{O}(|a|^3).$$

We are now ready to analyze (3.2):

$$(\lambda_0 - \lambda)a + \langle \psi_0, \gamma |a\psi_0 + h|^2 (a\psi_0 + h)\rangle = 0$$

where $h = h(a,\lambda)$, $a \in \mathbb{C}$, $|a| < \delta$

$a=0$ is a sln for all λ since $h(0,\lambda)=0$
We already knew about this solution since $F(0,\lambda)=0$ for all λ.

We now look for sln's $a\neq 0$. Divide by a

$$\lambda_0-\lambda+\langle \varphi_0, |a\varphi_0+h|^2(\varphi_0+\frac{h}{a})\rangle=0$$

Consider $a=|a|e^{i\theta}$ and use
$$h(a,\lambda)=h(|a|e^{i\theta},\lambda)=e^{i\theta}h(|a|,\lambda)$$
to get the equivalent equation:

$$(*)\quad \lambda_0-\lambda+\langle \varphi_0, (|a|\varphi_0+h(|a|,\lambda))^2(\varphi_0+\frac{h(|a|,\lambda)}{|a|})\rangle=0$$

Let
$$G(a,\lambda)=\begin{cases}\lambda_0-\lambda+\langle\varphi_0,(a\varphi_0+h(a,\lambda))^2(\varphi_0+\frac{h}{a})\rangle & \text{for } a\in\mathbb{R}\setminus\{0\}, |a|<\delta\\ 0 & \text{for } a=0\end{cases}$$

$(*)$ is the restriction to $a>0$ of
$$(**)\quad G(a,\lambda)=0$$
Now, one can check that $G:(-\delta,\delta)\times(-\delta_1,\delta_1)\to\mathbb{R}$ is C^1 and
$$G(0,\lambda_0)=0$$
$$\frac{\partial G}{\partial\lambda}(0,\lambda_0)=-1+\langle\varphi_0,\frac{3(a\varphi_0+h(a,\lambda))^2}{a}\cdot\frac{\partial h}{\partial\lambda}\rangle\Big|_{a=0}$$
$$=-1$$

By the implicit function theorem there is a unique curve of sln's for (**):

$$\lambda = \lambda(a) \quad a \in \mathbb{R},\ |a| < \tilde{\delta} \leq \delta$$

Translated back to the complex equation (*), we have a unique sln:

$$\lambda = \lambda(|a|) \quad a \in \mathbb{C} \quad |a| < \tilde{\delta}$$

and

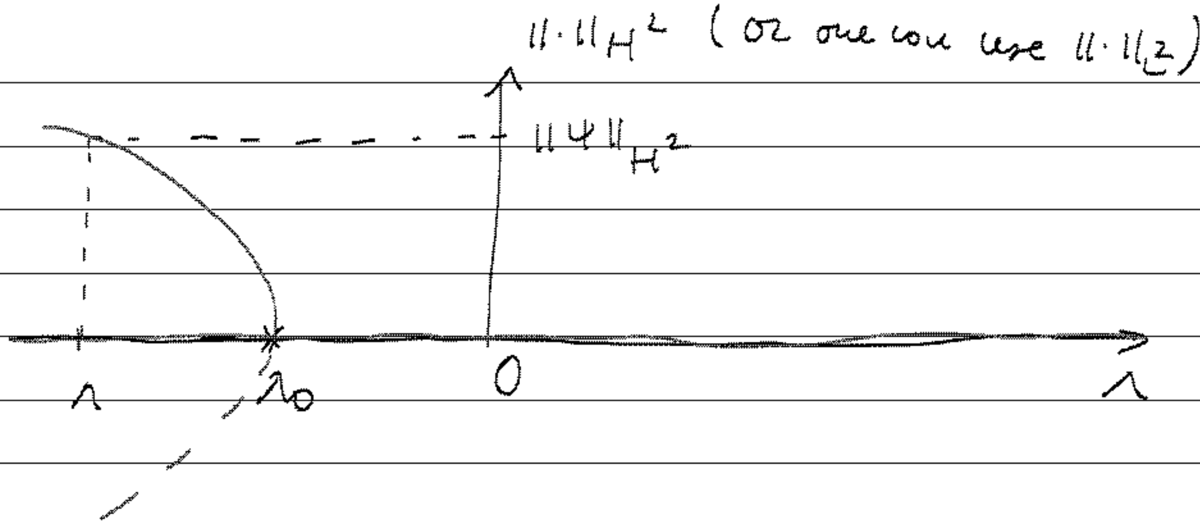
$$\psi = a\psi_0 + h(a, \lambda(|a|)) \quad a \in \mathbb{C}, |a| < \tilde{\delta}$$

the latter being the second branch of sln's of

$$F(\psi, \lambda) = 0 \quad \text{besides } \psi \equiv 0.$$

<u>Conclusion</u>: Eq (1) has a unique branch of nontrivial bound states bifurcating from a simple negative e-value of $-\Delta + V$.

<u>Remark</u> The type of bifurcation in (*) hence in the reduced eq. (3.2) is called a pitchfork bifurcation. We can draw the following bif. diagram:



where $\Psi = a\Psi_0 + u(a, \lambda(|a|))$
$\lambda = \lambda(|a|)$

References:

Pillet & Wayne: J. Diff. Eq 141(2): 310–326, 1997
Kirr & Zarnescu: Comm. Math. Phys. 272 (2007) no 2, 443–468.