Lecture 6 Math 555

Note Title 2/12/2008

2. Variational Methods

2.1 Euler-Lagrange Equations

2.2. An application to e-value problems and best constant in Poincaré Inequality

2.3. Sufficient conditions for existence of minimizers: coercivity and convexity.

2.4 Another application to e-value problems - Rellich generalized compactness

2.5. Nonlinear e-value problems, best constant in Gagliardo-Nirenberg inequality, concentration compactness.

2.7. Lyapunov functionals and orbital stability

2.1. Euler-Lagrange Equations.

X Banach space over reals, $U \subseteq X$ open

$F : U \longrightarrow \mathbb{R}$.

<u>Theorem</u> ($\mathcal{E}$-$\mathcal{L}$ without constraints)
If $u \in U$ is a local minima or maxima for F and the Gâteaux derivative of F at u in the direction $v \in X$ exists then

$$dF(u)[v] = 0$$

<u>Proof</u>: Assume $u \in U$ is a local minima
$\Rightarrow$ $\exists\, \varepsilon_0 > 0$ such that

$$F(u+\varepsilon v) \geq F(u) \quad \forall\, \varepsilon \in \mathbb{R},\ |\varepsilon| < \varepsilon_0$$

$$\Rightarrow \begin{cases} dF(u)v = \lim\limits_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \dfrac{F(u+\varepsilon v) - F(u)}{\varepsilon} \geq 0 \\ \\ dF(u)v = \lim\limits_{\substack{\varepsilon \to 0 \\ \varepsilon < 0}} \dfrac{F(u+\varepsilon v) - F(u)}{\varepsilon} \leq 0 \end{cases}$$

$\Rightarrow \ dF(u)[v] = 0$ QED

<u>Theorem</u> (Lagrange Multipliers) Assume that

1° $G : U \to \mathbb{R}$ is C^1;
2° $\mathcal{A} = \{ u \in U \mid G(u) = 0 \} \neq \phi$;
3° $u \in \mathcal{A}$ is a local minimizer for $F|_{\mathcal{A}}$;

4° $DG(u) \neq 0$;
5° F is Fréchet differentiable at u.

Then $\exists\ \lambda \in \mathbb{R}$ such that

$$DF(u)[v] = \lambda\, DG(u)[v] \text{ for all } v \in X$$

<u>Proof</u> Let $w \in X$ such that $DG(u)[w] \neq 0$
Define $j_v : \mathbb{R}^2 \to \mathbb{R} \quad v \in X$
$j_v(\tau, \theta) = G(u + \tau v + \theta w)$

Then $j_v(0,0) = G(u) = 0$ since $u \in \mathcal{A}$
$\frac{\partial j_v}{\partial \theta}(0,0) = DG(u)[w] \neq 0$

$\Rightarrow$ by implicit function theorem $\exists\ \delta > 0$ and

$\theta : (-\delta, \delta) \to \mathbb{R}$ a C^1 function such that

$j_v(\tau, \theta(\tau)) = 0 \ \forall \ \tau \in (-\delta, \delta)$ hence

$U + \tau V + \theta(\tau) W \in \mathcal{A} \quad \forall \ \tau \in (-\delta, \delta)$

Define $i : (-\delta, \delta) \longrightarrow \mathbb{R}$ by.

$$i(\tau) = F(U + \tau V + \theta(\tau) W)$$

Then 0 is a local minimiser for i. Since i is differentiable at 0 we have

$$0 = \frac{di}{d\tau}(0) = DF(U)[V + \theta'(0) W]$$

$$= DF(U)[V] + \theta'(0) DF(U)[W]$$

But from the implicit function theorem

$$\theta'(0) = -\frac{DG(U)[V]}{DG(U)[W]}$$

Hence

$$DF(U)[V] = \underbrace{\frac{DF(U)[W]}{DG(U)[W]}}_{\lambda \in \mathbb{R}} DG(U)[V] \quad \text{for any } V.$$

QED

2.2. An application to e-value problems and the best Poincare constant

Consider $\Omega \subseteq \mathbb{R}^n$ a bounded domain and

$$I : H_0^1(\Omega) \longrightarrow \mathbb{R}$$

$$I[u] = \int_\Omega |\nabla u(x)|^2 \, dx$$

<u>Theorem</u> If u_0 is a local minimizer of $I[u]$ subject to $\|u\|_{L^2(\Omega)} = 1$ then u_0 is a weak solution of:

$$-\Delta u_0 = \lambda u_0 \quad \text{for some } \lambda \in \mathbb{R}$$

<u>Proof</u> Use Lagrange multiplier theorem:

$$dI[u_0]v = \lambda DG(u_0)v \quad \forall v \in H_0^1$$

$$\Leftrightarrow \int_\Omega \nabla u_0 \cdot \nabla v \, dx = \lambda \int_\Omega u_0 \cdot v \, dx \quad \forall v \in H_0^1$$

but this is the def of weak sln's of $-\Delta u_0 = \lambda u_0$

<u>Theorem</u> The problem:

$$\inf_{\substack{v \in H_0^1 \\ \|v\|_{L^2} = 1}} I[v] = \lambda$$

verifies:

(i) It has a minimizer v_0, unique modulo multiplication by $e^{i\theta}$ $0 \le \theta < 2\pi$.
(ii) λ is the lowest e-value of $-\Delta$ on Ω with zero bdry condition. v_0 is the corresponding e-vector
(iii) $v_0 \in C^2(\Omega) \cap C(\bar{\Omega})$
(iv) λ is a simple e-value and v_0 can be chosen strictly positive.

<u>Proof</u> (i) Clearly $0 \le \lambda < \infty$. Let $\{v_k\} \subseteq H_0^1(\Omega)$ be a minimizing sequence

$$\Rightarrow I[v_k] \xrightarrow{k \to \infty} \lambda$$

$$\Rightarrow \int_\Omega |\nabla v_k|^2 dx \xrightarrow{k \to \infty} \lambda.$$

$$\Rightarrow \|\nabla v_k\|_{L^2} \le \sqrt{\lambda} + 1 \text{ for } k \text{ suff large}$$

$$\Rightarrow \|v_k\|_{H_0^1} = \|v_k\|_{L^2} + \|\nabla v_k\|_{L^2} = 1 + \|\nabla v_k\|_{L^2} \le 2 + \sqrt{\lambda}$$

$\Rightarrow$ $\{v_k\}$ bounded in H_0^1
$H_0^1(\Omega)$ reflexive Banach space $\Big\}$ $\exists \{v_{k_j}\} \subseteq \{v_k\}$

$v_{k_j} \xrightarrow{j \to \infty} v_0$ in $H_0^1(\Omega)$

$$\Rightarrow \int_\Omega \nabla v_{k_j} \nabla v \, dx + \int_\Omega v_{k_j} v \, dx \xrightarrow{j \to \infty}$$

$$\int_\Omega \nabla v_0 \nabla v \, dx + \int_\Omega v_0 v \, dx \quad \forall v \in H_0^1$$

$v_{k_j} \longrightarrow v_0$ in $H_0^1(\Omega)$
$H_0^1 \underset{\text{compact}}{\hookrightarrow} L^2$ $\Big\}$ $\Rightarrow v_{k_j} \longrightarrow v_0$ in L^2
$\Rightarrow \|v_0\|_{L^2} = 1$

(Rellich compactness theorem, relies on Ω bounded)

and from above we get:

$$\int_\Omega \nabla v_{k_j} \cdot \nabla v \, dx \longrightarrow \int_\Omega \nabla v_0 \cdot \nabla v \, dx$$

Now:

$$\int_\Omega |\nabla v_0|^2 dx \le \liminf \int_\Omega |\nabla v_{k_j}|^2 dx = \lambda$$

Indeed by Cauchy-Schwarz

$$\left(\int_\Omega \nabla v_{k_j} \cdot \nabla v_0 \, dx\right)^2 \le \int_\Omega |\nabla v_{k_j}|^2 dx \int_\Omega |\nabla v_0|^2 dx$$

Apply lim inf, note that LHS is convergent:

$$\left(\int |\nabla v_0|^2 dx\right)^2 \le \int_\Omega |\nabla v_0|^2 dx \cdot \liminf \int_\Omega |\nabla v_{k_j}|^2 dx$$

So $I[v_0] \le \lambda$,

$\|v_0\|_{L^2} = 1 \Rightarrow \underset{\text{inf/minimum}}{\lambda} \le I[v_0]$

$\Bigg\} \Rightarrow \lambda = I[v_0]$

$\Rightarrow v_0$ is a minimizer. Part (i) is done

(ii) The fact that λ is an e-value with corresponding e-vector v_0 follows from previous theorem.

If λ_1 is another e-value with corresponding e-vector v_1, normalized to $\|v_1\|_{L^2} = 1$, we have

$$\int_\Omega \nabla v_1 \cdot \nabla v \, dx = \lambda_1 \int_\Omega v_1 \cdot v \, dx \quad \forall v \in H_0^1$$

$\Rightarrow \quad I[v_1] = \lambda_1$ (make $v = v_1$ above).

$\|v_1\|_{L^2} = 1$

$\Rightarrow \lambda \leq \lambda_1$.

(iii) Standard regularity theory for elliptic e-value problems:

For example to obtain $v_0 \in C^2(\Omega)$ consider $\Omega_1 \subset\subset \Omega$, i.e. Ω_1 open and $\overline{\Omega_1} \subset$ Interior of Ω.

Let χ_1 be a C^∞ version of the characteristic function of Ω_1; i.e $\chi_1 \in C^\infty(\mathbb{R}^n)$

$$\chi_1(x) = 1 \text{ for } x \in \Omega_1; \ \operatorname{supp} \chi_1 \in \Omega.$$

Then $\chi_1 v_0$ satisfies in the weak sense

$$-\Delta(\chi_1 v_0) = \lambda(\chi_1 v_0) + \nabla(\nabla\chi_1 \cdot v_0)$$

on $\mathbb{R}^n$. Using Fourier Transform

$$|\xi|^2 \widehat{\chi_1 v_0}(\xi) = \lambda \widehat{\chi_1 v_0}(\xi) + \underbrace{\cdots}_{\in L^2_\xi(\mathbb{R}^n)}$$

$$\Rightarrow (1+|\xi|^2) \widehat{\chi_1 v_0}(\xi) = \underbrace{(\lambda+1) \widehat{\chi_1 v_0}(\xi) + \cdots}_{\in L^2(\mathbb{R}^n)}$$

$$\Rightarrow (1+|\xi|^2) \widehat{\chi_1 v_0}(\xi) \in L^2(\mathbb{R}^n)$$

$$\Rightarrow \chi_1 v_0 \in H^2(\mathbb{R}^n)$$

$$\Rightarrow 2\nabla\chi_1 \cdot \chi_1 v_0 \in H^2(\mathbb{R}^n) \text{ and } \chi_1^2 v_0 \in H^2$$

$$\Rightarrow -\Delta(\chi_1^2 v_0) = \lambda(\chi_1^2 v_0) + \underbrace{\nabla(2\nabla\chi_1 \cdot \chi_1 v_0)}_{\in H^1}$$

Repeat the argument above $\Rightarrow$ $\chi_1^2 v_0 \in H^3(\mathbb{R}^n)$

Inductively $\chi_1^{m-1} v_0 \in H^m(\mathbb{R}^n)$

by choosing m large enough so that

$(m-2)\cdot 2 > n$ we get.

$x_1^{m-1} v_0 \in C^2(\mathbb{R}^n)$.

$\Rightarrow v_0 \in C^2(\Omega_1)$.

Since $\Omega_1 \subset\subset \Omega$ arbitrary we get
$v_0 \in C^2(\Omega)$.

(iv) Since $|\nabla |v|| \leq |\nabla v| \quad \forall v \in H_0^1(\Omega)$
we deduce that
$\tilde{v}_0 = |v_0| \in H_0^1(\Omega)$ is also a minimizer

Obviously $\tilde{v}_0 \geq 0$ a.e. $\Rightarrow$ $\tilde{v}_0 \geq 0$
By (iii) $\tilde{v}_0 \in C^2(\Omega)$ on Ω.

If $x_0 \in \Omega$ is such that $\tilde{v}_0(x_0) = 0$ then
$x_0 \in B(x_0, \varepsilon) \subseteq \Omega$ is a minima for $\tilde{v}_0$ in
contradiction to the strong maximum principle
for $-\tilde{v}_0$ on $B(x_0,\varepsilon)$. $\Rightarrow \tilde{v}_0 > 0$ on Ω.

Now we have that the minimizer can be chosen such that $u_0 > 0$ on Ω.

Let u_1 be another minimizer $\Rightarrow$ u_1 is another e-vector for λ (in the weak sense) $\Rightarrow$ $\mathrm{Re}\, u_1$, $\mathrm{Im}\, u_1$ are e-vectors.

Denote $u_2 = \mathrm{Re}\, u_1$, $u_3 = \mathrm{Im}\, u_1$

<u>Claim</u> $\exists\, c_1, c_2 \in \mathbb{R}$ such that $u_2 = c_1 u_0$, $u_3 = c_2 u_0$

Assume not, orthogonalize u_2 and u_0 and renormalize u_2 to have $\|u_2\|_{L^2} = 1$. Then

$$\int_\Omega u_0 u_2 \, dx = 0 \Rightarrow u_2 \text{ changes sign on } \Omega$$

and since u_2 remains an e-vector $\Rightarrow |u_2|$ is a minimizer attaining its minimum on Ω in contradiction with the above argument.

In conclusion $u_1 = (c_1 + i c_2) u_0$ and since $\int_\Omega |u_1|^2 dx = 1 = \int_\Omega |u_0|^2 dx \Rightarrow |c_1 + i c_2| = 1$.

Now if u_1 would be another e-vector of for λ then normalize it such that $\|u_1\|_{L^2} = 1$ hence u_1 will be a minimizer and consequently

$u_1 = c\,u_0$ for some $c \in \mathbb{C}$.

Combining this with results from the theory of compact operators which imply that the invariant subspace of $-\Delta$ (on a bounded Ω with 0 bdry condition) corresponding to λ is finite dimensional on which $-\Delta$ is represented by a self adjoint matrix we get that the invariant space for λ is one dimensional.

The theorem is now completely proven.

Remark

$$\lambda = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^2} = 1}} \int_\Omega |\nabla u|^2 \, dx = \inf_{\substack{u \in H_0^1 \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

So the lowest e-value of $-\Delta$ on Ω with zero boundary condition is also the best constant for Poincare ineq:

$$\|u\|^2_{L^2(\Omega)} \leq C_1 \|\nabla u\|^2_{L^2(\Omega)} \qquad u \in H_0^1(\Omega)$$