Lecture 7 Math 555

Note Title 2/24/2008

Summary:

2.3. Sufficient conditions for existence of minimizers: coercivity and convexity.

2.4 Another application to e-value problems - Rellich generalized compactness

2.3 Coercivity and convexity

X reflexive Banach space over reals

$F : X \to \mathbb{R}$, $G : X \to \mathbb{R}$

$\mathcal{A} = \{x \in X : G(x) = 0\} \neq \phi$

Def F is coercive on $\mathcal{A}$ if for any $\{x_n\} \subseteq \mathcal{A}$ such that $\|x_n\| \to \infty$ we have

$$F(x_n) \longrightarrow +\infty$$

Def F is convex iff $\forall\ 0 \le t \le 1$:

$$F((1-t)x + ty) \le (1-t)F(x) + tF(y).$$

Theorem 1: If $F: X \to \mathbb{R}$ is continuous then the following are equivalent.

1. F is convex
2. $\forall x \in X \ \exists h \in X^*$ such that

$$F(y) - F(x) \geq \langle h, y-x \rangle \quad \forall y \in X$$

Remark 1 The continuity hypothesis is not needed when X is finite dimensional. In this case $F: X \to \mathbb{R}$ convex implies F continuous. In infinite dimensional Banach spaces F convex and bounded from above in a neighborhood of a point implies continuity.

Remark 2 If $\forall x \in X$ and $y \in X$, $d^2F(x)[y,y]$ exists and $d^2F(x)[y,y] \geq 0$ then F is convex.

Recall:

$$d^2F(x)[y,y] = \lim_{\varepsilon \to 0} \frac{dF(x+\varepsilon y)[y] - dF(x)[y]}{\varepsilon}$$

See Handout for proofs and further discussions!

Corollary: If F is continuous and convex then F weakly lower semicontinuous.

Proof Let $y_n \rightharpoonup x$, i.e.

$$\langle h, y_n \rangle \to \langle h, x \rangle \quad \forall h \in X^*.$$

We need to show

$$F(x) \leq \liminf_{n \to \infty} F(y_n)$$

F continuous and convex $\Rightarrow$ $\exists\, h \in X^*$ such that

$$F(y_n) - F(x) \geq \langle h, y_n - x \rangle$$

$$\Rightarrow \liminf_{n\to\infty} \left[F(y_n) - F(x)\right] \geq \liminf_{n\to\infty} \langle h, y_n - x\rangle$$

$$= 0$$

$$\Rightarrow \liminf_{n\to\infty} F(y_n) \geq F(x) \quad \text{QED.}$$

<u>Theorem 2</u> (existence of minimizers)
Assume F is continuous, convex on X and coercive on $\mathcal{A}$, and G is weakly continuous.
Then $F|_{\mathcal{A}}$ has a global minimizer.

<u>Proof</u> Let $\lambda = \inf_{x\in\mathcal{A}} F(x) < +\infty$ (since $\mathcal{A} \neq \emptyset$)

and let $\{x_n\}$ be a minimizing sequence:

$$\lim_{n\to\infty} F(x_n) = \lambda, \quad x_n \in \mathcal{A} \quad \forall n \in \mathbb{N}.$$

$\{x_n\}$ is bounded, otherwise $\exists \{x_{n_k}\}$ such that $\|x_{n_k}\| \to \infty$. But then, by coercivity

$$F(x_{n_k}) \to +\infty \neq \lambda \quad \text{contradiction}$$

$$\left.\begin{array}{l}\{x_n\} \text{ bounded} \\ X \text{ reflexive}\end{array}\right\} \Rightarrow \exists \{x_{n_k}\} \text{ weakly convergent } x_{n_k} \rightharpoonup x \in X.$$

G weakly continuous $\Rightarrow G(x_{n_k}) \to G(x) \Rightarrow x \in \mathcal{A}$

F continuous and convex $\Rightarrow$ F weakly lower semicontinuous

$$\Rightarrow F(x) \leq \liminf_{n\to\infty} F(x_{n_k}) = \lambda.$$

$\Rightarrow x$ is a minimizer for $F|_{\mathcal{A}}$ Q.E.D.

<u>Remark 3</u> If F is strictly convex and $\mathcal{A}$ convex then the minimizer is unique.

Indeed if x_1, x_2 are minimizers then

$$F\left(\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2\right) < \tfrac{1}{2}F(x_1) + \tfrac{1}{2}F(x_2) = \lambda$$

in contradiction with $\lambda = \inf_{x\in\mathcal{A}} F(x)$.

2.4 Another application to e-value problems: Rellich generalized compactness

Consider the e-value problem:

(1) $\quad -\Delta u(x) + V(x)u(x) = \lambda u(x) \quad x \in \mathbb{R}^n$

we are looking for weak solutions $u \in H^1(\mathbb{R}^n)$.

Assume:

(H1) $V : \mathbb{R}^n \to \mathbb{R}, \ V \in L^\infty(\mathbb{R}^n)$

(H2) $\lim_{|x| \to \infty} V(x) = 0$

Consider the related minimizing problem:

(2) $$\mu = \inf_{\substack{u \in H^1 \\ \|u\|_{L^2} = 1}} \int_{\mathbb{R}^n} |\nabla u|^2 + V|u|^2 \, dx$$

<u>Remark</u> If u is a minimizer for (2) then u is a solution for (1) with $\lambda = \mu$.

Indeed for

$$F = \int_{\mathbb{R}^n} |\nabla u|^2(x)\,dx + \int_{\mathbb{R}^n} V(x)\, |u|^2(x)\,dx$$

$$G = \int_{\mathbb{R}^n} |u|^2(x)\,dx - 1$$

If u is a minimizer for F subject to $G(u)=0$ then by $\mathcal{E}$-$\mathcal{L}$ Theorem, $\exists \lambda \in \mathbb{R}$

$$dF(u)[v] = \lambda\, DG(u)[v] \quad \forall v \in H^1$$

$$\Leftrightarrow (*) \int_{\mathbb{R}^n} \nabla u \cdot \nabla v\, dx + \int_{\mathbb{R}^n} V(x)\, u\, v\, dx$$

$$= \lambda \int_{\mathbb{R}^n} u\, v\, dx \quad \forall v \in H^1$$

$\Leftrightarrow$ $-\Delta u + Vu = \lambda u$ in the weak sense

Moreover plugging $v=u$ in $(*)$ we get

$$\mu = F(u) = \lambda(G(u)+1) = \lambda \Rightarrow \mu = \lambda.$$

Theorem If (H1), (H2) are satisfied and $\exists\, v \in H^1$, $\|v\|_{L^2} = 1$ such that $F(v) < 0$ then problem (2) has a minimizer.

Note Since $d^2 F(v)[v,v] = F(v) < 0$ we deduce that F is not convex. Moreover $G: H^1 \to L^2$ $G(v) = \|v\|_{L^2}^2 - 1$ is not compact. Both will be circumvented by the following:

Lemma (generalized Rellich compactness)
If (H1), (H2) are satisfied and $v_k \rightharpoonup v$ in H^1 then
$$\int_{\mathbb{R}^n} V(x)\,|v_k(x)|^2\,dx \longrightarrow \int_{\mathbb{R}^n} V(x)\,|v(x)|^2\,dx$$

Proof of Lemma $v_k \rightharpoonup v \Rightarrow \exists M > 0 : \|v_k\|_{H^1} \le M\ \forall k$ and $\|v\|_{H^1} \le M$. Hence $\|v_k\|^2, \|v\|^2 \le M^2\ \forall k$.

Let $\tilde{V}_k = \int_{\mathbb{R}^n} V(x)\,|v_k(x)|^2\,dx$

$$\tilde{V} = \int_{\mathbb{R}^n} V(x)\,|v(x)|^2\,dx$$

We claim $\lim\limits_{k\to\infty} \tilde{V}_k = \tilde{V}$.

Indeed for any $\varepsilon > 0$ we can choose $R > 0$ such that, by (H2),

$$|V(x)| \leq \frac{\varepsilon}{4M^2} \quad \forall |x| > R$$

Since $u_k \rightharpoonup u$ in $H^1(B(0,R))$
$H^1(B(0,R)) \underset{\text{compact}}{\hookrightarrow} L^2(B(0,R))$ $\Bigg\} \Rightarrow u_k \xrightarrow{L^2(B(0,R))} u$

$\Rightarrow \exists K > 0 : \| |u_k|^2 - |u|^2 \|_{L^1(B(0,R))} < \frac{\varepsilon}{2\|V\|_{L^\infty}} \quad \forall k > K$

Hence for $k > K$ we have:

$$|\tilde{V}_k - \tilde{V}| \leq \int_{B(0,R)} |V(x)| \, \big| |u_k|^2 - |u|^2 \big| \, dx$$

$$+ \int_{\mathbb{R}^n \setminus B(0,R)} |V(x)| \left(|u_k|^2 + |u|^2 \right) dx$$

$$\leq \|V\|_{L^\infty} \| |u_k|^2 - |u|^2 \|_{L^1(B(0,R))}$$

$$+ \frac{\varepsilon}{4M^2} \left(\|u_k\|_{L^2}^2 + \|u\|_{L^2}^2 \right) < \varepsilon$$

Proof of Theorem :

$\mu \leq F(U) < 0$

Let $\{U_k\}$ be a minimising sequence. Since F is coercive on $\|V\|_{L^2} = 1 \Rightarrow \{U_k\}$ is bounded in H^1

$$\left.\begin{array}{l}\{U_k\} \text{ bounded in } H^1 \\ H^1(\mathbb{R}^n) \text{ reflexive}\end{array}\right\} \Rightarrow \exists \{U_{n_k}\} \subseteq \{U_n\}$$
$$U_{n_k} \rightharpoonup U_0 \text{ in } H^1$$

By Rellich generalised compactness

$$\int_{\mathbb{R}^n} V(x) |U_{n_k}|^2 dx \xrightarrow{k \to \infty} \int_{\mathbb{R}^n} V(x) |U_0|^2 dx$$

Hence

$$\mu = \liminf_{k \to \infty} F(U_{n_k}) = \liminf_{k \to \infty} \int_{\mathbb{R}^n} |\nabla U_{n_k}|^2 dx + \int_{\mathbb{R}^n} V |U_0|^2 dx$$
$$\geq \int_{\mathbb{R}} |\nabla U_0|^2 dx + \int_{\mathbb{R}^n} V |U_0|^2 dx = F(U_0)$$

where we used the convexity, hence weakly lower semicontinuity, of

$$F_1 : H^1 \to \mathbb{R}$$

$$F_1[v] = \int_{\mathbb{R}^n} |\nabla v|^2 dx$$

It remains to show $\|v_0\|_{L^2} = 1$. Assume not.

By convexity of $\tilde{G} : H^1 \to \mathbb{R}$ $\tilde{G}(v) = \|v\|^2_{L^2}$ we have that

$$\|v_0\|^2_{L^2} \leq \liminf_{k \to \infty} \|v_{n_k}\|^2_{L^2} = 1$$

In addition, from

$$F(v_0) \leq \mu < 0$$

we deduce $v_0 \neq 0 \Rightarrow \|v_0\|_{L^2} > 0$.

Now, since $0 < \|v_0\|^2_{L^2} < 1$, we have:

$$0 > \mu \geq F(v_0) > \frac{F(v_0)}{\|v_0\|^2_{L^2}} = F(\tilde{v}_0)$$

where $\tilde{v}_0 = v_0 / \|v_0\|_{L^2} \Rightarrow \|\tilde{v}_0\|_{L^2} = 1$

This contradicts $F(\tilde{v}_0) \geq \mu = \inf_{\substack{v \in H^1 \\ \|v\|_{L^2}=1}} F(v)$

So v_0 is a minimizer for (2). QED

<u>Corollary</u> If V satisfies (H1) and (H2) and $\exists\, v \in H^1$, $\|v\|_{L^2} = 1$ such that

$$\int_{\mathbb{R}^n} |\nabla v|^2 dx + \int_{\mathbb{R}^n} V(x) |v|^2 dx < 0$$

then $-\Delta + V : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$ has a lowest e-value μ which is simple with corresponding e-vector $v_0 \in H^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ which can be chosen such that $v_0 > 0$.

<u>Proof</u> The fact that μ solving (2) is the lowest e-value follows directly from the previous remark and theorem.

The fact that v_0 can be chosen such that $v_0 \geq 0$ follows from the fact that $|\nabla |v_0|| \leq |\nabla v_0|$ a.e.

hence

$$F(|v_0|) \le F(v_0)$$

For the strict since $v_0 > 0$ and the fact that μ is simple we first need to prove the regularity of v_0: $v_0 \in H^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

For any H^1 solution of.

$$-\Delta v = \mu v - V(x) v.$$

we have

$$(**) \quad \mathcal{F}^{-1}\left[(1+|\xi|^2)\mathcal{F}v(\xi)\right] = (1+\mu)v - V(x)v$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denotes the Fourier Transform.

By Sobolev imbedding $v \in H^1(\mathbb{R}^n) \Rightarrow v \in L^p(\mathbb{R}^n)$ where

$$2 \le p \le \frac{2n}{n-2} \quad \text{for } n > 2$$

$$2 \le p < \infty \quad \text{for } n = 1,2$$

Consequently $(1+\mu)v - V(x)v \in L^p(\mathbb{R}^n)$ and by $(**)$:

$$\mathcal{F}^{-1}\left[(1+|\xi|^2)\,\mathcal{F}u(\xi)\right] \in L^p(\mathbb{R}^n)$$

which by Mihlin Multiplier Theorem (see page 16) implies:

$(***)$ $\quad u \in W^{2,p}(\mathbb{R}^n)$.

If $n=1,2$ $(***)$ holds for all $2 \le p < \infty$. By choosing p large enough so that

$$2p > n$$

we get via Sobolev imbedding

$$u \in W^{2,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$$

If $n > 2$ $(***)$ hold for $2 \le p \le \frac{2n}{n-2}$.

and by Sobolev imbedding

$$u \in L^q(\mathbb{R}^n) \text{ where } 2 \le q \le \infty \text{ if } n < 6$$

$$\text{or } 2 \le q \le \frac{2n}{n-6} \text{ if } n > 6.$$

By repeating the above argument with p replaced by q eventually we obtain

$$u \in L^q(\mathbb{R}^n) \cap C(\mathbb{R}^n) \quad 2 \le q \le \infty.$$

So $v_0 \in H^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is now proven and using Harnack inequality for $v_0 \geq 0$ solution in $H^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ of

$$\Delta v_0 - (V(x) - \mu) v_0 = 0$$

we obtain for any $\Omega \subset \mathbb{R}^n$ open and bdd $\exists\, C_\Omega$ such that

$$\sup_{x \in \Omega} v_0(x) \leq C_\Omega \inf_{x \in \Omega} v_0(x).$$

If there $\exists\, x_0 \in \mathbb{R}^n$ such that $v_0(x_0) = 0 \leq v_0(x)\ \forall x$ we would get

$v_0(x) = 0$ on any $\Omega \subseteq \mathbb{R}^n$ open and bdd

and by continuity $v_0(x) = 0$ on $\mathbb{R}^n$. contradiction with $\cancel{f_0}(v_0) = \mu < 0$

Finally if v_1 is another e-vector for μ then so is $\bar{v}_1$ and $\operatorname{Re} v_1$ and $\operatorname{Im} v_1$.

If v_0 and $\operatorname{Re} v_1$ are linearly independent then $\{v_0, \operatorname{Re} v_1\}$ can be orthonormalized in L^2 to $\{v_0, v_2\}$ where v_2 is still an e-vector for μ and $\|v_2\|_{L^2} = 1$

$$\text{From } \left.\begin{array}{l} \displaystyle\int_{\mathbb{R}^n} v_0(x)\, v_2(x)\, dx = 0 \\ v_0 \geq 0 \end{array}\right\} \Rightarrow v_2 \text{ changes}$$

and since v_2 is still a solution of (1) $\Rightarrow v_2 \in \mathbb{C}(\mathbb{R}^n)$.
$\Rightarrow \exists\, x_0$ such that $v_2(x_0) = 0$.

Then $|v_2|$ is another minimizer for (2) hence e-vector for μ hence $|v_2| > 0 \;\forall x \in \mathbb{R}^n$
Contradiction.

Consequently v_0 and $\operatorname{Re} v_1$ or similarly v_0 and $\operatorname{Im} v_1$ are linearly dependent hence $\exists\, c \in \mathbb{C}$ such that $v_1 = c v_0$. So the e-vector space of μ is generated by v_0. Since $-\Delta + V$ is s.a. on L^2 it cannot have generalized e-vectors so μ is simple!

The corollary is finished. QED.

__Mihlin Multiplier Theorem__ Denote

$H^{s,p}(\mathbb{R}^n) = \{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2} \mathcal{F}\cdot u(\xi)] \in L^p \}$

Then for s integer and $1 < p < \infty$

$$H^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$$

__Proof__ For $s = 2$ and $1 < p < \infty$ we have for $u \in H^{2,p}$:

$$\left| \langle v, \mathcal{F}^{-1}[(1+|\xi|^2)\mathcal{F}u(\xi)] \rangle \right| < \infty \quad \forall\, v \in L^{p'}$$

$$\frac{1}{p} + \frac{1}{p'} = 1$$

But $\langle v, \mathcal{F}^{-1}[(1+|\xi|^2)\mathcal{F}u(\xi)] \rangle =$

$$\langle \mathcal{F}v, (1+|\xi|^2)\mathcal{F}u(\xi) \rangle$$

and since $|\xi_i \xi_j| \le \frac{\xi_i^2 + \xi_j^2}{2} \le 1 + |\xi|^2$

we get $\forall v \in L^{p'}$

$$\left| \langle \mathcal{F}v, \xi_i \xi_j \mathcal{F}u(\xi) \rangle \right| < \infty.$$

$$\shortparallel$$

$$\left| \langle v, -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} u(x) \rangle \right| < \infty$$

$$\Rightarrow \frac{\partial^2}{\partial x_i \partial x_j} u \in L^p(\mathbb{R}^n)$$

Similar argument for $\xi_i \xi_j$ replaced by $1, \xi_i$ lead to $u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbb{R}^n) \Rightarrow u \in W^{2,p}(\mathbb{R}^n)$

Now if $u \in W^{2,p}(\mathbb{R}^n) \Rightarrow 1 - \Delta u \in L^p(\mathbb{R}^n)$

$$\Rightarrow \left| \langle v, (1-\Delta) u \rangle \right| < \infty \quad \forall v \in L^{p'}(\mathbb{R}^n)$$

$$\shortparallel$$

$$\left| \langle \mathcal{F}v, (1+|\xi|^2) \mathcal{F}u \rangle \right|$$

$$\shortparallel$$

$$\left| \langle v, \mathcal{F}^{-1}(1+|\xi|^2) \mathcal{F}u \rangle \right|$$

So $|\langle v, \mathcal{F}^{-1}(1+|\xi|^2)\mathcal{F}u(\xi)\rangle| < \infty$
for all $v \in L^{p'} \Rightarrow \mathcal{F}^{-1}(1+|\xi|^2)\mathcal{F}u \in L^p(\mathbb{R}^n)$
$\Rightarrow u \in H^{s,p}(\mathbb{R}^n)$.

Similar arguments hold for $s = 1, 2, 3, \ldots$
for $s = 0$ is trivial and $s = -1, -2, \ldots$ can be obtained by duality:

$$(H^{s,p}(\mathbb{R}^n))^* = H^{-s,p}(\mathbb{R}^n)$$
$$(W^{s,p}(\mathbb{R}^n))^* = W^{-s,p}(\mathbb{R}^n).$$ QED.

<u>Remark</u> It should be possible to replace (H1) by the weaker assumption:

(H1)' $V : \mathbb{R}^n \to \mathbb{R}$, $V \in L^p(\mathbb{R}^n)$ with $p \geq 1$ and $p > n/2$.

throughout this lecture. Check!