Lecture 8 Math 555

Note Title 3/3/2008

Concentration Compactness and nonlinear e-value problems

In both previous applications of the variational method to e-value problems we needed a sort of compactness to show that the minimizer is in the admissable set. On bounded domains we used:

Theorem (Rellich Compactness) If $\Omega \subset \mathbb{R}^N$ is bounded then

$$H_0^1(\Omega) \underset{\text{compact}}{\hookrightarrow} L^p(\Omega)$$

for
$$\begin{cases} 2 \le p < \dfrac{2N}{N-2} & \text{if } N > 2 \\ 2 \le p < \infty & \text{if } N = 2 \\ 2 \le p \le \infty & \text{if } N = 1 \end{cases}$$

On unbounded domains we used generalized Rellich compactness which can be rewritten (check!):

<u>Theorem</u> (Generalized Rellich Compactness)
Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be bounded and assume $u_n(x) \to 0$ as $|x| \to \infty$ <u>uniformly</u> in n. Then $\exists \{u_{n_k}\} \subset \{u_n\}$ and $u \in H^1$ such that:

$$u_{n_k} \xrightarrow{L^p} u \quad \text{for } 2 < p < \frac{2N}{N-2} \text{ for } N \geq 2$$

$$(2 < p \leq \infty \text{ for } N = 1).$$

<u>Proof</u> $\left.\begin{array}{l} H^1(\mathbb{R}^N) \text{ reflexive} \\ \{u_n\} \text{ bdd in } H^1 \end{array}\right\} \Rightarrow \exists \{u_{n_k}\} \subset \{u_n\}$

$u_{n_k} \rightharpoonup u$ in H^1

Since $H^1(B(0,M)) \underset{\text{compact}}{\hookrightarrow} L^1(B(0,M))$

$\Rightarrow u_{n_k} \to u$ in $L^1(B(0,M))$

$\Rightarrow \exists\, u_{n_{k_j}} \to u$ a.e on $B(0,M)$

By letting $M \to \infty$ and choosing a diagonal sequence of u_{n_k} we get a subsequence of $\{u_n\}$ redenoted by $\{u_{n_k}\}$ such that

$u_{n_k} \to u$ a.e on $\mathbb{R}^N$ and $u_{n_k} \rightharpoonup u$

Fix $\varepsilon > 0$ and $R > 0$ (R to be choosen later)

$$\|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} \leq \|u_{n_k} - u\|_{L^p(B(0,R))}$$

$$+ \|u_{n_k} - u\|_{L^p(|x|>R)}$$

$$\overset{\text{Hölder}}{\leq} \|u_{n_k} - u\|_{L^p(B(0,R))} +$$

$$+ \|u_{n_k} - u\|^{(p-2)/p}_{L^\infty(|x|>R)} \underbrace{\|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}}_{\text{bounded}}$$

Now choose R sufficiently large :

$$\underbrace{\|u_{n_k} - u\|^{(p-2)/p}_{L^\infty(|x|>R)}}_{\to 0 \text{ unif as } |x| \to \infty} \underbrace{\|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}}_{\text{unif bdd}} < \frac{\varepsilon}{2}$$

for all k

Then choose k large enough such that

$$\|u_{n_k} - u\|_{L^p(B(0,R))} < \frac{\varepsilon}{2}$$

Since again $H^1(B(0,R)) \xrightarrow[\text{compact}]{} L^p(B(0,R))$

and $u_{n_k} \xrightarrow{H^1} u$ implies $u_{n_k} \longrightarrow u$ in $L^p(B(0,R))$

We get $\forall \varepsilon > 0$ $\exists K$ such that for $k > K$

$$\|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} < \varepsilon \qquad QED$$

<u>Concentration compactness</u> is a method that investigates what happens in the previous theorem when uniform convergence to zero at infinity is removed.

<u>Counterexample</u> $u_0(x) = \begin{cases} 1-|x| & 0 \le |x| < 1 \\ 0 & \text{otherwise} \end{cases}$

$u_n(x) = u_0(x-n)$. Then $\{u_n\} \subset H^1(\mathbb{R})$ bounded
$u_n \rightharpoonup 0$ in H^1 but $u_{n_k} \not\to 0$ in $L^p(\mathbb{R})$
regardless of the choice of subsequence u_{n_k} or the choice of p, $2 \le p \le \infty$.

Concentration compactness shows that a bounded sequence $\{u_n\}$ in $H^1(\mathbb{R}^N)$ is in one and only one of the following situations.

1° By choosing appropriate translations $y_n \in \mathbb{R}^N$ $\{u_n(\cdot - y_n)\}$ has a convergent subsequence in L^p, $2 \le p < 2N/(N-2)$ for $N \ge 2$, $2 \le p \le \infty$ for $N = 1$

2° $\{u_n\}$ has a subsequence convergent to zero in L^p, $2 < p < (2N)/(N-2)$

3° There is a "splitting": a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and sequences $\{v_k\}, \{w_k\} \subset H^1$ such that

$$\|u_{n_k} - v_k - w_k\|_{L^p} \xrightarrow{k\to\infty} 0 \quad 2 \le p < \frac{2N}{N-2}$$

$$(\text{or } 2 \le p < \infty \text{ if } N=1)$$

$$\operatorname{dist}(\operatorname{supp}(v_k), \operatorname{supp}(w_k)) \xrightarrow{k\to\infty} \infty$$

$\{v_k\}$ is in situation 1°.

For more precise statements see the handout!