Lecture 9 Math 555

Note Title 3/7/2008

Application to e-value problems for nonlinear Schrödinger equation. Existence of bound states.

Consider $\alpha > 0$ and:

(1) $$-\Delta u - |u|^{\alpha} u = \lambda u \qquad u \in H^1(\mathbb{R}^N), \ \lambda \in \mathbb{R}$$

and the "energy" functional:

$$\mathcal{H} : H^1(\mathbb{R}^N) \longrightarrow \mathbb{R}$$

$$\mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2} dx$$

Since $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ where

$$\begin{cases} 2 \le p \le \frac{2N}{N-2} & \text{if } N > 2 \\ 2 \le p < \infty & \text{if } N = 2 \\ 2 \le p \le \infty & \text{if } N = 1 \end{cases}$$

a sufficient condition for $\mathcal{H}$ to be well defined is

$$\alpha + 2 \le \frac{2N}{N-2} \Leftrightarrow \alpha \le \frac{4}{N-2} \quad \text{if } N > 2$$

From now one we assume

1° $0 < \alpha \le \frac{4}{N-2}$ if $N > 2$, $0 < \alpha < \infty$ for $N = 1, 2$

Theorem 1 If the problem:

$$(2) \qquad I_\nu = \inf_{\substack{v \in H^1 \\ \|v\|^2_{L^2} = \nu}} \mathcal{H}(v)$$

has a minimizer v_0, then v_0 is a solution of the e-value problem (1) for some $\lambda \in \mathbb{R}$

Proof: Direct conseq of Euler-Lagrange eq-

Theorem 2: Fix $0 < \alpha \leq \frac{4}{N-2}$. Let

$$(2) \quad I_\mu = \inf_{\substack{\|u\|_{L^2}^2 = \mu \\ u \in H^1(\mathbb{R}^N)}} \mathcal{H}(u) = \inf \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{\alpha+2} \int_{\mathbb{R}^N} |u|^{\alpha+2} dx$$

Then:

(i) (Subcritical case) for $0 < \alpha < \frac{4}{N}$ we have

$$-\infty < I_\mu < 0$$

(ii) (critical case) for $\alpha = \frac{4}{N}$ there $\exists \mu_0 > 0$ such that

$$I_\mu = 0 \quad \text{for } 0 < \mu \leq \mu_0$$

$$I_\mu = -\infty \quad \text{for } \mu > \mu_0$$

(iii) (supercritical case) for $\frac{4}{N} < \alpha \leq \frac{4}{N-2}$

$$I_\mu = -\infty$$

<u>Proof</u> (iii) Fix $\mu > 0$ and $u \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|u\|_{L^2}^2 = \mu.$$

Consider $u_L(x) = L^{N/2} u(Lx) \qquad L > 0$

Then $\|u_L\|_{L^2} = \|u\|_{L^2}$

$$\|\nabla u_L\|_{L^2} = L \, \|\nabla u\|_{L^2}$$

$$\|u_L\|_{L^{\alpha+2}} = L^{\frac{N\alpha}{2(\alpha+2)}} \|u\|_{L^{\alpha+2}}.$$

Hence

$$\mathcal{H}(u_L) = \frac{L^2}{2} \overbrace{\int_{\mathbb{R}^N} |\nabla u|^2 dx}^{>0 \text{ fixed}} - \frac{L^{\frac{N\alpha}{2}}}{\alpha+2} \overbrace{\int_{\mathbb{R}^N} |u|^{\alpha+2} dx}^{>0 \text{ fixed}}$$

If $\alpha > \frac{4}{N}$ then $\frac{N\alpha}{2} > 2$ and

$$\mathcal{H}(u_L) \longrightarrow -\infty \text{ as } L \to \infty$$

$$\Rightarrow \quad I_\mu = -\infty$$

(i) Sobolev - Gagliardo - Nirenberg.

$$\|U\|_{L^p} \leq C_{N,p} \|\nabla U\|_{L^2}^{N(\frac{1}{2}-\frac{1}{p})} \|U\|_{L^2}^{1-N(\frac{1}{2}-\frac{1}{p})}$$

for $\begin{cases} 2 \leq p \leq \frac{2N}{N-2} & \text{if } N > 2 \\ 2 \leq p < \infty & \text{if } N = 1, 2 \end{cases}$

Use it in $\mathcal{H}(U)$ for $p = \alpha + 2$:

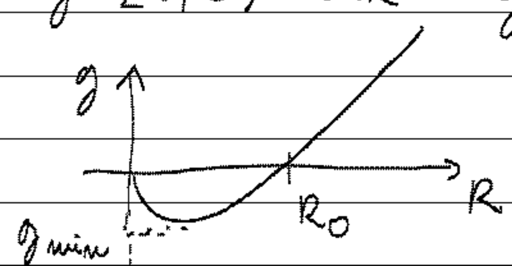
$$\mathcal{H}(U) \geq \frac{1}{2} \|\nabla U\|_{L^2}^2 - \frac{C_{N,p}^{\alpha+2}}{\alpha+2} \|\nabla U\|_{L^2}^{N\frac{\alpha}{2}} \mu^b$$

where $b = \left[1 - N\left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\right](\alpha+2)/2$

So:

$\mathcal{H}(U) \geq g(\|\nabla U\|_{L^2})$ where

$$g : [0, \infty) \to \mathbb{R} \qquad g(R) = \frac{1}{2} R^2 - \frac{C_{N,p}^{\alpha+2}}{\alpha+2} \mu^b R^{N\frac{\alpha}{2}}$$



Hence

$$I_N \geq g_{min} > -\infty.$$

For $0 > I_N$ use the scaling from part (iii):

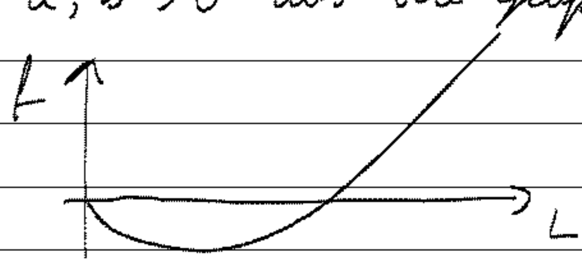
- fix $u \in C_c^\infty(\mathbb{R}^N)$ $\|u\|_{L^2}^2 = N$

- $u_L(x) = L^{N/2} u(Lx)$ $L > 0$

$\Rightarrow \|u_L(x)\|_{L^2}^2 = N$ and.

$$\mathcal{H}(u_L) = \frac{L^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{L^{\frac{N\alpha}{2}}}{\alpha+2} \int_{\mathbb{R}^N} |u|^{\alpha+2} dx$$

Since $f: (0,\infty) \to \mathbb{R}$ $f(L) = L^2 a - L^{\frac{N\alpha}{2}} b$
with $a, b > 0$ has the graph:



By picking $L > 0$ small enough $\Rightarrow \mathcal{H}(u_L) < 0$
$\Rightarrow I_N < 0$

Part (ii) Sobolev - Gagliardo - Nirenberg:

$$\mathcal{H}(u) \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 \left(1 - 2\,\frac{C_{N,\alpha+2}^{\alpha+2}}{\alpha+2}\, \mu^{\frac{2}{N}}\right)$$

where $\alpha = \frac{4}{N}$.

Now for $\mu \leq \left(\frac{\alpha+2}{2} \cdot \frac{1}{C_{N,\alpha+2}^{\alpha+2}}\right)^{N/2}$

we have $\mathcal{H}(u) \geq 0 \Rightarrow I_\mu \geq 0$. To show $I_\mu = 0$ use again the scaling:

$$u_L(x) = L^{N/2} u(Lx) \quad u \in C_c^\infty(\mathbb{R}^N),\ \|u\|_{L^2}^2 = \mu \text{ fixed}$$

$$\mathcal{H}(u_L) = L^2 \underbrace{\left(\frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2}\right)}_{\geq 0.}, \quad L > 0$$

$\mathcal{H}(u_L) \to 0$ as $L \to 0 \Rightarrow I_\mu = 0$

(*) For $\mu > \left(\frac{\alpha+2}{2} \cdot \frac{1}{C_{N,\alpha+2}^{\alpha+2}}\right)^{N/2}$ where

We assume that the constant coming from the Sobolev - Gagliardo - Nirenberg inequality has been choosen to be the best:

$$C_N = \sup \left\{ \frac{\|u\|_{L^{4/N+2}}}{\|\nabla u\|_{L^2}^{N/(N+2)} \cdot \|u\|_{L^2}^{2/(N+2)}} \;\middle|\; \begin{array}{l} u \in H^1(\mathbb{R}^N) \\ u \neq 0 \end{array} \right\}.$$

Consequently for any $\theta < 1$ $\exists\, u_\theta \in H^1(\mathbb{R}^N)$

$$(**) \quad \|u_\theta\|_{L^{4/N+2}} \geq \theta C_N \, \|\nabla u_\theta\|_{L^2}^{N/(N+2)} \, \|u_\theta\|_{L^2}^{2/(N+2)}$$

By changing u_θ with $c \cdot u_\theta$, $c > 0$ we can assume $\|u_\theta\|_{L^2}^2 = \mu$ satisfies $(*)$. Note that $u_\theta \to c u_\theta$ does not change $(**)$.

Then

$$\mathcal{H}(u_\theta) \leq \frac{1}{2} \|\nabla u_\theta\|_{L^2}^2 \underbrace{\left(1 - \frac{2 (\theta C_N)^{\frac{2(2+N)}{N}}}{\frac{2(2+N)}{N}} \mu^{2/N} \right)}_{<0 \text{ for } \theta \text{ sufficiently close to } 1.}$$

$\Rightarrow \mathcal{H}(u_\theta) < 0.$

Now for $u_{L,\theta}(x) = L^{N/2} u_\theta(Lx) \quad L > 0$

we have $\|u_{L,\theta}(x)\|^2_{L^2} = \mu$ and.

$$\mathcal{H}(u_{L,\theta}) = L^2 \underbrace{\mathcal{H}(u_\theta)}_{<0} \xrightarrow{L\to\infty} -\infty$$

$\Rightarrow I_\mu = -\infty$ for μ satisfying $(*)$. QED

Theorem 3 (existence of minimizers in the subcritical case).

If $0 < \alpha < \frac{4}{N}$ then

(2) $$I_\mu = \inf_{\substack{u \in H^1 \\ \|u\|^2_{L^2} = \mu}} \mathcal{H}(u)$$

has a minimizer.

Corollary: The minimizers of (2) are $C^\infty(\mathbb{R}^N)$ are unique up to translations and rotations and if v_0 is a minimizer, $\exists\, \theta \in [0, 2\pi)$ such that $e^{i\theta} v_0$ is strictly positive.

<u>Proof of Corollary</u> If v_0 is a minimizer then by theorem 1 $\exists\, \lambda \in \mathbb{R}$ such that:

(3) $$-\Delta v_0 - |v_0|^{\alpha} v_0 = \lambda v_0$$

$v_0 \in C^{\infty}(\mathbb{R}^N)$ follows from using Fourier Transform and Mihlin multiplier theorem, see Lecture 7.

If v_0 is a minimizer then $|v_0|$ is a minimizer because $|\nabla |v_0|| \leq |\nabla v_0|$ a.e. Hence $\exists\, \tilde{\lambda}$ such that

(4) $$-\Delta |v_0| - |v_0|^{\alpha} |v_0| = \tilde{\lambda} |v_0|$$

Multiply (3) by $\overline{v_0}$ and (4) by $|v_0|$, integrate over $\mathbb{R}^N$ and subtract:

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} (\nabla |v_0|)^2 dx = (\lambda - \tilde{\lambda}) \mu$$

But LHS is zero because $\mathcal{H}(v_0) = \mathcal{H}(|v_0|) = I_\mu$ $\Rightarrow \lambda = \tilde{\lambda}$.

Hence both v_0 and $|v_0|$ are e-vectors corresponding to e-value λ of the linear operator:

$$-\Delta + V(x), \quad V(x) = -|U_0|^2$$

From Lecture 7 we know that $-\Delta + V(x)$ has a lowest e-value ν, simple, with e-vector v which can be choosen stricbly positive. Now $\nu = \lambda$ otherwise we must have

$$\int v(x)\, |U_0(x)|\, dx = 0$$

which is impossible since $v(x) > 0$ and $|U_0(x)| \geq 0$, $U_0(x) \not\equiv 0$ a.e. Hence $|U_0(x)| = c\, v(x) \Rightarrow |U_0(x)| > 0$ and $U_0(x) = d\, v(x) \Rightarrow U_0(x) = e^{i\theta} |U_0(x)|$.

Uniqueness of $|U_0(x)|$ up to translation is not trivial. Can make up a project for this class. QED.

<u>Proof of Theorem 3</u> Let $\{U_n\} \subseteq H^1(\mathbb{R}^N)$, $\|U_n\|_{L^2}^2 = \nu$ be a minimizing sequence. Since $I_\nu < 0$ we have $\mathcal{H}(U_n) < 0$ for n sufficiently large and via Sobolev - Gagliardo - Nirenberg:

$$g(\|\nabla U_n\|_{L^2}) \leq \mathcal{H}(U_n) < 0 \text{ for } n \text{ suff large}$$

$\Rightarrow \|\nabla v_n\|_{L^2} \leq R_0$ for n sufficiently large, see page 5.

Therefore $\{v_n\}$ is bdd in $H^1(\mathbb{R}^N)$.
Moreover

$$\mathcal{H}(v_n) \leq \frac{I_\mu}{2} < 0 \text{ for } n \text{ suff large}$$

$$\Leftrightarrow \frac{1}{2}\|\nabla v_n\|^2_{L^2} - \frac{1}{\alpha+2}\|v\|^{\alpha+2}_{L^{\alpha+2}} \leq \frac{I_\mu}{2}$$

$\Rightarrow \|v_n\|_{L^{\alpha+2}} \geq \text{const} > 0$ for n suff large.

Since $2 \leq \alpha+2 < \frac{2N}{N-2}$ vanishing (case 2°)

in concentration compactness theorem (see Lecture 8) cannot happen.

Assume that splitting (case 3°) happens. Then $\exists \{v_{n_k}\} \subseteq \{v_n\}$ and $\{v_k\} \subseteq H^1$, $\{w_k\} \subseteq H^1$ such that

$$\|v_k\|^2_{L^2} \to \nu < \mu, \quad \|w_k\|^2_{L^2} \to \mu - \nu$$

$$\int_{\mathbb{R}^N} |v_{n_k}|^p dx - \int_{\mathbb{R}^N} |v_k|^p dx - \int_{\mathbb{R}^N} |w_k|^p dx \xrightarrow{k\to\infty} 0$$

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N} |\nabla v_{n_k}|^2 dx - \int_{\mathbb{R}^N} |\nabla v_k|^2 - \int |\nabla w_k|^2 dx \geq 0$$

where $2 \leq p < \frac{2N}{N-2}$ so $p = \alpha+2 < \frac{4}{N}+2$ is allowed.

Then we have for any $\varepsilon > 0$ and k sufficiently large:

$$I_\mu + \varepsilon \geq \mathcal{H}(u_{m_k}) + \frac{\varepsilon}{2}$$

$$= \frac{1}{2} \|\nabla u_{m_k}\|_{L^2}^2 - \frac{1}{\alpha+2} \|u_{m_k}\|_{L^{\alpha+2}}^{\alpha+2} + \frac{\varepsilon}{2}$$

(#) $$\geq \frac{1}{2} \|\nabla v_k\|_{L^2}^2 + \frac{1}{2} \|\nabla w_k\|_{L^2}^2 - \frac{1}{\alpha+2} \|v_k\|_{L^{\alpha+2}}^{\alpha+2} - \frac{1}{\alpha+2} \|w_k\|_{L^{\alpha+2}}^{\alpha+2} + \frac{\varepsilon}{4}$$

$$= \mathcal{H}(v_k) + \mathcal{H}(w_k) + \frac{\varepsilon}{4}$$

$$\geq I_\nu + I_{\mu-\nu}, \quad \nu < \mu.$$

On the other hand; for $\theta > 1$ we have

$$\mathcal{H}(\theta u) = \theta^2 \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{\theta^{\alpha+2}}{\alpha+2} \|u\|_{L^{\alpha+2}}^{\alpha+2}$$

$$< \theta^2 \left(\frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{\alpha+2} \|u\|_{L^{\alpha+2}}^{\alpha+2} \right)$$

$$= \theta^2 \mathcal{H}(u)$$

Therefore $I_{\theta^2 \mu} < \theta^2 I_\mu \quad \forall\, \theta > 1$ and for $\nu < \mu$ we have

$$I_\mu = I_{\frac{\mu}{\nu}\cdot\nu} < \frac{\mu}{\nu} I_\nu = \frac{\mu-\nu+\nu}{\nu} I_\nu$$

$$= \frac{\mu-\nu}{\nu} I_\nu + I_\nu = \frac{\mu-\nu}{\nu} I_{\frac{\nu}{\mu-\nu}(\mu-\nu)} + I_\nu$$

$$\leq \frac{\mu-\nu}{\nu} \cdot \frac{\nu}{\mu-\nu} I_{\mu-\nu} + I_\nu$$

$$= I_{\mu-\nu} + I_\nu$$

all in all $I_\mu < I_\nu + I_{\mu-\nu}$ for $\nu < \mu$ which contradicts (#).

Hence splitting cannot happen either and we can conclude that case 1° in concentration compactness theorem happens, i.e.:

$\exists \{u_{n_k}\} \subseteq \{u_n\}$ and $y_k \in \mathbb{R}^N$, $u_0 \in H^1(\mathbb{R}^N)$ such that

$$v_k = u_{n_k}(\cdot - y_k) \longrightarrow u_0$$

$$v_k \xrightarrow{L^p} u_0 \qquad 2 \leq p < \frac{2N}{N-2}.$$

In particular for $p = 2 \Rightarrow \|u_0\|^2_{L^2} = \mu$

for $p = \alpha+2 \Rightarrow \|v_k\|_{L^{\alpha+2}} \to \|u_0\|_{L^{\alpha+2}}$

By weak lower semicontinuity of

$$U \longmapsto \frac{1}{2} \|\nabla U\|_{L^2}^2$$

we conclude that

$\mathcal{H}(U_0) \leq \liminf_{k \to \infty} \mathcal{H}(V_k) = I_\mu$ since

$V_k = U_{n_k}(\cdot - y_k)$ is also minimizing.

$\Rightarrow$ $\mathcal{H}(U_0) = I_\mu$ and U_0 is a minimizer QED.