Nonlinear Phenomena in Partial Differential Equations

Eduard Kirr
University of Illinois at Urbana-Champaign

Urbana, Feb. 2009

Collaborators: P.G. Kevrekidis (UMass), D. Pelinovsky (McMaster), E. Shlizerman (Weizmann) and M.I. Weinstein (Columbia)

Problem: Find (orbitally) stable boundstates for the NLS eq:

$$i\partial_t u(t,x) = (-\Delta_x + V(x))u - |u|^2 u \qquad (1)$$

- $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$;

- $V : \mathbb{R}^n \to \mathbb{R}$, in L^q, $2 \le q \le \infty$ and

$$V(-x_1, x_2, \ldots, x_n) = V(x_1, x_2, \ldots, x_n).$$

More general nonlinearities: Instead of $-|u|^2$ use

$$g(x,|u|) = \sigma(x)|u|^2 + o(|u|^2) \quad \text{or} \quad g(|u|) = \int_{\mathbb{R}^n} K(x-y)|u(y)|^2 dy.$$

Applications: Nonlinear Optics, Water Waves, Statistical Physics in particular Bose-Einstein Condensates.

Nonlinear Boundstates

Solutions of the form:

$$u(t,x) = e^{-i\Omega t}\psi_\Omega(x), \qquad \Omega \in \mathbb{R}, \quad \psi_\Omega \in H^1(\mathbb{R}^n),$$

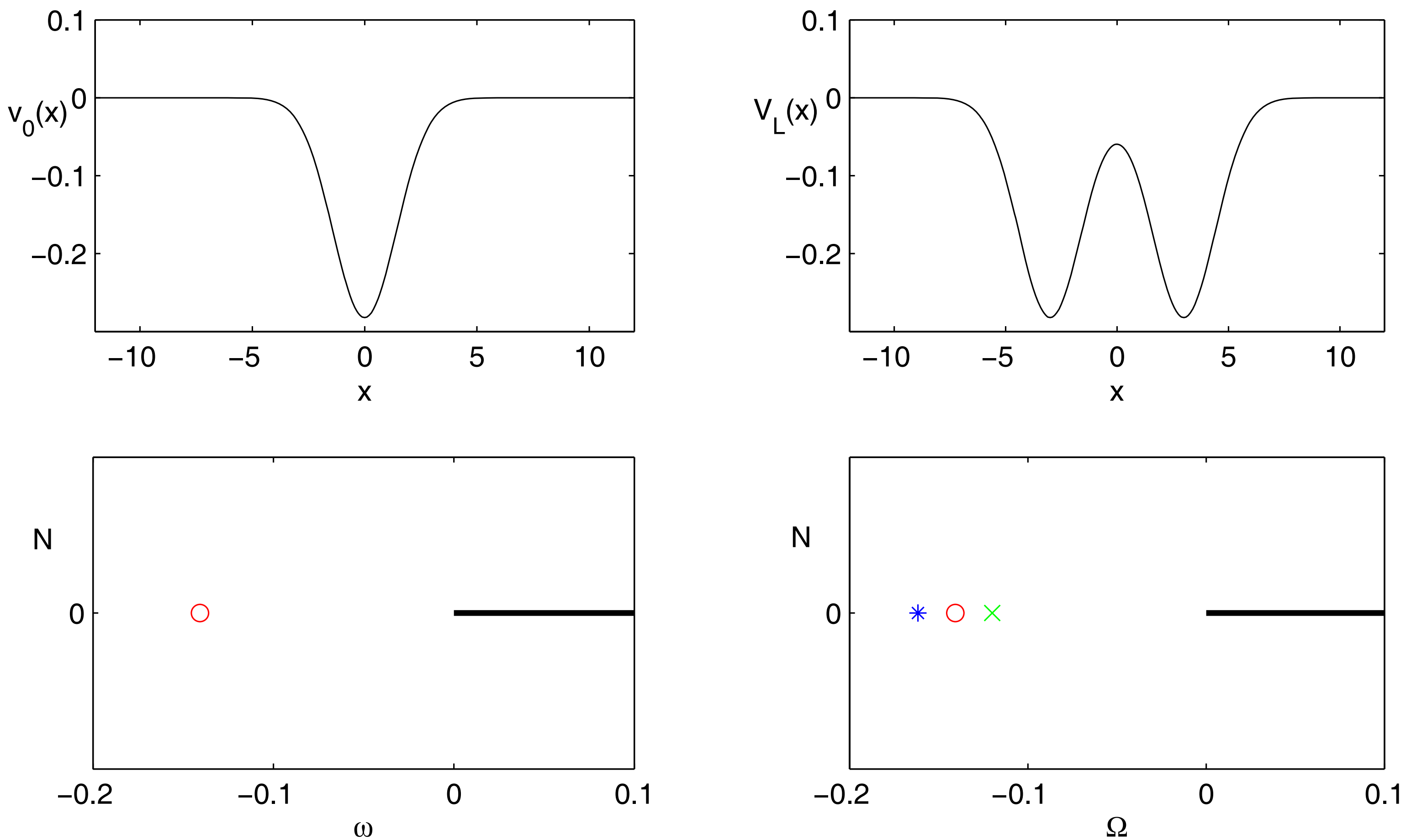
are called boundstates. Hence ψ_Ω satisfies in the weak sense:

$$(-\Delta + V)\psi_\Omega - |\psi_\Omega|^2\psi_\Omega = \Omega\psi_\Omega. \tag{2}$$

Spectral Assumption: $-\Delta + V$ in L^2 has the two lowest e-values, Ω_0, Ω_1, *negative, simple*, and *relatively close* to each other.

Example: Double well potentials.

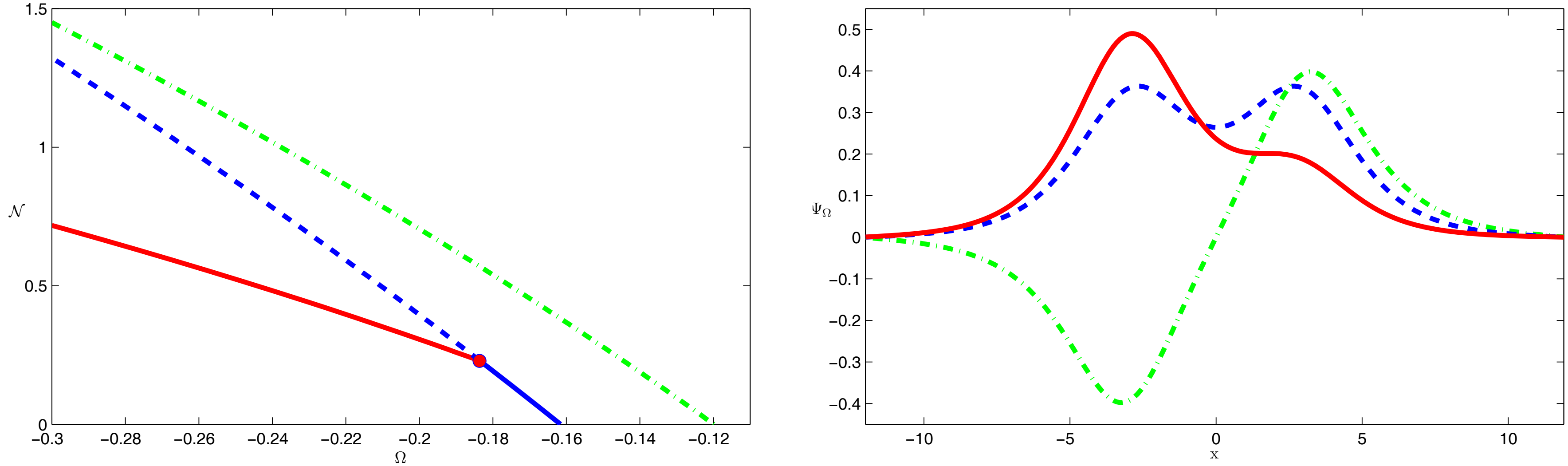
Single well   Double well



$$(-\Delta + V)\psi_\Omega - |\psi_\Omega|^2\psi_\Omega = \Omega\psi_\Omega,\ \mathcal{N} = \|\psi_\Omega\|_{L^2}^2$$

Bifurcation diagram

Boundstates at $\mathcal{N} = 0.5$



The bifurcations diagram has been proven in Kevrekidis-Kirr-Shlizerman-Weinstein, SIAM J. Math. Anal. 2008. Before it was known that:

- One branch of solutions to (2) bifurcates from Ω_0. It is formed by solutions which are even in x_1 and are orbitally stable (W, R-W, GSS late '80s).

- A second branch bifurcates from Ω_1. It is formed by solutions to (2) which are odd in x_1.

- If $K(x) = -1/|x|$ and $n = 3$ then for sufficiently large L^2 norm groundstates are neither even nor odd in x_1 (AFGST '02). Their variational approach can be generalized to the cubic NLS for $n = 1$.

New Result

The even branch bifurcates again at $(\Omega_{cr}, N_{cr}), N_{cr} > 0$ into an asymmetric branch which is orbitally stable (groundstates) and an even branch which becomes unstable.

Heuristical Ansatz:

$$\psi_\Omega = c_0\psi_0 + c_1\psi_1 + neglect, \qquad c_0,\ c_1 \in \mathbb{C},$$

where: ψ_0 is the real valued e-vector of $-\Delta + V$ corresponding to e-value Ω_0. Note that ψ_0 is necessarily even.

ψ_1 is the real valued e-vector of $-\Delta + V$ corresponding to e-value Ω_1. Note that ψ_1 is odd if V is a double well potential.

Plug in (2):

$$(-\Delta + V)\psi_\Omega - \psi_\Omega\bar{\psi_\Omega}\psi_\Omega = \Omega\psi_\Omega,$$

and project onto ψ_0 and ψ_1 :

$$
\begin{aligned}
(\Omega_0 - \Omega)c_0 + a_{0000}|c_0|^2 c_0 + (a_{0110} + a_{0011})|c_1|^2 c_0 + a_{0101} c_1^2 \bar{c_0} &= 0 \\
(\Omega_1 - \Omega)c_1 + a_{1111}|c_1|^2 c_1 + (a_{1001} + a_{1100})|c_0|^2 c_1 + a_{1010} c_0^2 \bar{c_1} &= 0
\end{aligned}
$$

where $a_{klmn} = -\langle \psi_k, \psi_l \bar{\psi_m} \psi_n \rangle$.

Even branch: $c_1 = 0,\ |c_0| = \sqrt{(\Omega - \Omega_0)/a_{0000}},\ \psi_\Omega = c_0(\Omega)\psi_0.$

Odd branch: $c_0 = 0,\ |c_1| = \sqrt{(\Omega - \Omega_1)/a_{1111}},\ \psi_\Omega = c_1(\Omega)\psi_1.$

For any other sln's write: $c_0 = \rho_0 e^{i\theta_0},\ c_1 = \rho_1 e^{i\theta_1},\ \Delta\theta = \theta_1 - \theta_0$

$$
\begin{aligned}
\rho_0[\Omega_0 - \Omega + a_{0000}\rho_0^2 + (a_{0110} + a_{0011}(1 + e^{i2\Delta\theta}))\rho_1^2] &= 0 \\
\rho_1[\Omega_1 - \Omega + a_{1111}\rho_1^2 + (a_{1001} + a_{1100}(1 + e^{-i2\Delta\theta}))\rho_0^2] &= 0.
\end{aligned}
$$

Take imaginary parts: $\sin(2\Delta\theta) = 0$ if $a_{0011} \neq 0$.

For $\Delta\theta \in \{0, \pi\}$:

$$\begin{aligned} \rho_0[\Omega_0 - \Omega + a_{0000}\rho_0^2 + (a_{0110} + 2a_{0011})\rho_1^2] &= 0 \\ \rho_1[\Omega_1 - \Omega + a_{1111}\rho_1^2 + (a_{1001} + 2a_{1100})\rho_0^2] &= 0; \end{aligned}$$

For $\Delta\theta \in \{\pi/2, 3\pi/2\}$:

$$\begin{aligned} \rho_0[\Omega_0 - \Omega + a_{0000}\rho_0^2 + a_{0110}\rho_1^2] &= 0 \\ \rho_1[\Omega_1 - \Omega + a_{1111}\rho_1^2 + a_{1001}\rho_0^2] &= 0. \end{aligned}$$

Both can be analyzed and are universal unfolding of bifurcations with two degrees of freedom and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry. If

$$a_{0000} - a_{1001} - 2a_{1100} > 0, \quad a_{1111} - a_{0110} - 2a_{0011} > 0 \quad (3)$$

$$a_{1111} - a_{0110} > 0, \quad a_{1001} - a_{0000} > 0 \quad (4)$$

then the second system has no new solutions, while the first one has an asymmetric branch of solutions bifurcating from the even one.

FROM:

M. GOLUBITSKY & D. SCHAEFFER

"SINGULARITIES AND GROUPS IN BIFURCATION THEORY"

VOL. I

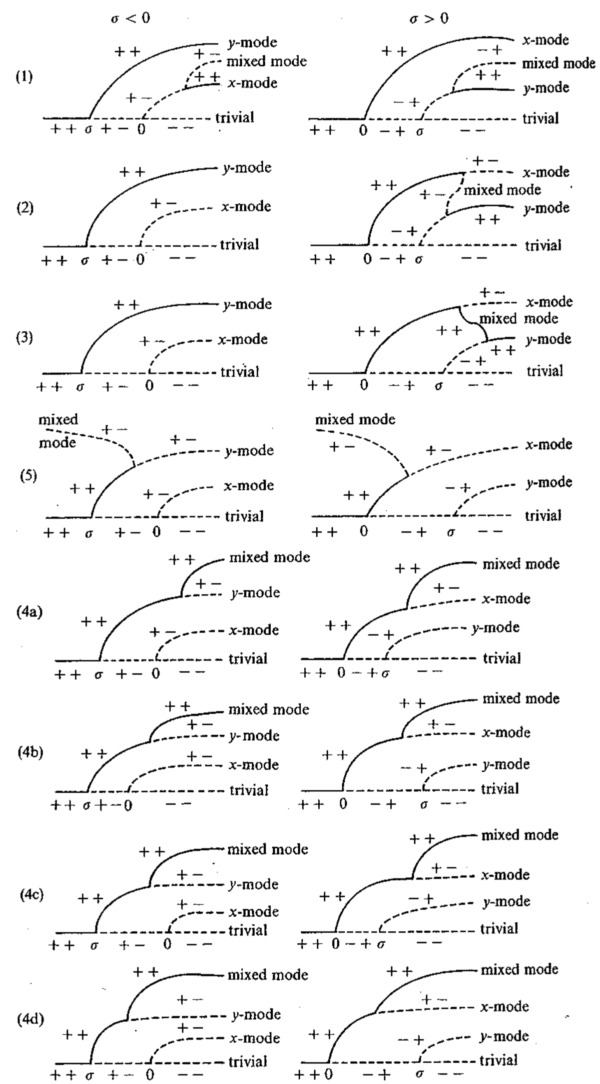


Figure 4.3. Persistent perturbations of (4.6). Numbers refer to regions in Fig. 4.1.

Lyapunov-Schmidt like reduction

$$(-\Delta + V)\psi_\Omega - |\psi_\Omega|^2\psi_\Omega = \Omega\psi_\Omega. \tag{5}$$

$$\begin{aligned} \psi_\Omega &= c_0\psi_0 + c_1\psi_1 + \eta \\ \eta &\in \{\psi_0, \psi_1\}^\perp \subset L^2 \end{aligned}$$

Denote by $P_\perp$ the projection onto $\{\psi_0, \psi_1\}^\perp$, then (5) is equivalent with:

$$\langle \psi_0, (-\Delta + V - \Omega)\psi_\Omega - |\psi_\Omega|^2\psi_\Omega \rangle = 0 \tag{6}$$
$$\langle \psi_1, (-\Delta + V - \Omega)\psi_\Omega - |\psi_\Omega|^2\psi_\Omega \rangle = 0 \tag{7}$$
$$P_\perp[(-\Delta + V - \Omega)\psi_\Omega - |\psi_\Omega|^2\psi_\Omega] = 0 \tag{8}$$

The real Fréchet derivative w.r.t η of (8) at $(c_0, c_1, \eta) = (0, 0, 0)$, $(-\Delta + V - \Omega)P_\perp$, is invertible for Ω close to Ω_0 or Ω_1. Implicit function theorem gives $\eta = \eta(c_0, c_1, \Omega)$ which plugged in (6-7) leads to:

$$
\begin{aligned}
(\Omega_0 - \Omega)c_0 + a_{0000}|c_0|^2 c_0 + (a_{0110} + a_{0011})|c_1|^2 c_0 + a_{0101} c_1^2 \bar{c}_0 & \\
+ g_0(c_0, c_1, \Omega) &= 0 \\
(\Omega_1 - \Omega)c_1 + a_{1111}|c_1|^2 c_1 + (a_{1001} + a_{1100})|c_0|^2 c_1 + a_{1010} c_0^2 \bar{c}_1 & \\
+ g_1(c_0, c_1, \Omega) &= 0
\end{aligned}
$$

where

$$
g_j(c_0, c_1, \Omega) \leq C(|c_0| + |c_1|)^5, \qquad j = 0, 1.
$$

Although g_0, g_1 are higher order they can still influence the bifurcation diagram:

– why $c_0 = 0$ or $c_1 = 0$ remain solutions?

– why the imaginary part of the above eqn's is satisfied iff the phase difference between c_0 and c_1 is an integer multiple of $\pi/2$?

The symmetries of the problem

$$F[\psi] = (-\Delta + V - \Omega)\psi - |\psi|^2\psi$$

is equivariant with respect to:

– rotations, $\psi \mapsto e^{i\theta}\psi, \qquad 0 \leq \theta < 2\pi$

– complex conjugation, $\psi \mapsto \bar{\psi}$,

– reflections w.r.t $x_1 = 0$, $\psi(x_1, x_2, \ldots, x_n) \mapsto \psi(-x_1, x_2, \ldots, x_n)$

This is an $O(2) \times \mathbb{Z}_2$ equivariance. The reduced system will be $O(2) \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ equivariant: $[c_0, c_1] \mapsto [e^{i\theta}c_0, e^{i\theta}c_1]$, $[c_0, c_1] \mapsto [\bar{c_0}, \bar{c_1}]$, $[c_0, c_1] \mapsto [\alpha c_0, \beta c_1]$, $\alpha = \pm 1$, $\beta = \pm 1$.

Consequently

$$g_j = c_j \sum_{k+l+m+n\geq 4,\ k-l+m-n=0,\ m+n=even} b^j_{klmn}(\Omega) c_0^k \bar{c_0}^l c_1^m \bar{c_1}^n$$

or for $c_j = \rho_j e^{i\theta_j},\ \Delta\theta = \theta_1 - \theta_0$:

$$g_j = c_j \sum_{\tilde{k}+\tilde{m}\geq 2,\ p\in\mathbb{Z}} b^j_{kmp}(\Omega) e^{ip2\Delta\phi} \rho_0^{2\tilde{k}} \rho_1^{2\tilde{m}}.$$

Now, from the reduced eqn's, $\sin(2\Delta\phi) = 0$, and the reduced system collapses on two real systems with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry in which the higher order terms $g_j,\ j = 0, 1$ play no qualitative role (singularity theory).

A less perturbative look

$$F[\psi,\Omega] = (-\Delta + V)\psi - |\psi|^2\psi - \Omega\psi = 0$$

Real Fréchet derivative w.r.t ψ at a real valued ϕ, and Ω in the direction $u + iv$:

$$DF_\psi(\phi,\Omega)[u+iv] = \mathcal{L}[u+iv] = \begin{bmatrix} \Re\mathcal{L} \\ \Im\mathcal{L} \end{bmatrix} = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

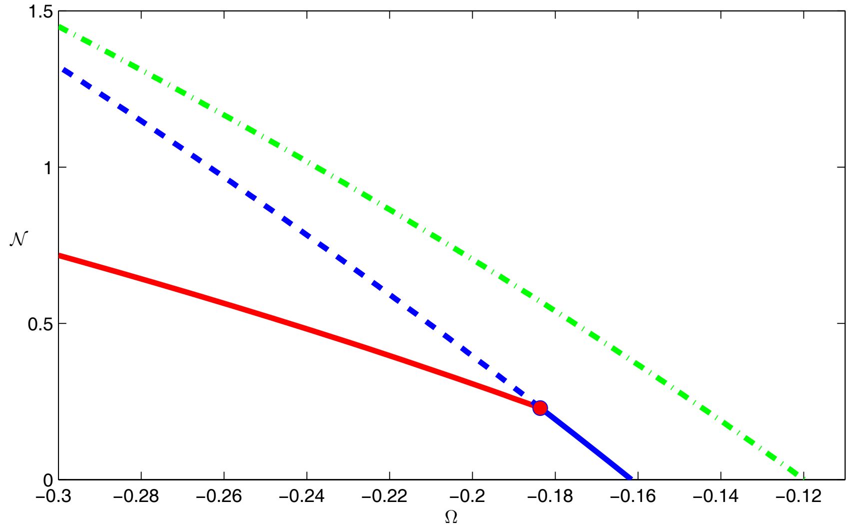
where

$$\begin{aligned} L_-[v] &= (-\Delta + V - \Omega)v - |\phi|^2 v \\ L_+[u] &= (-\Delta + V - \Omega)u - 3|\phi|^2 u \end{aligned}$$

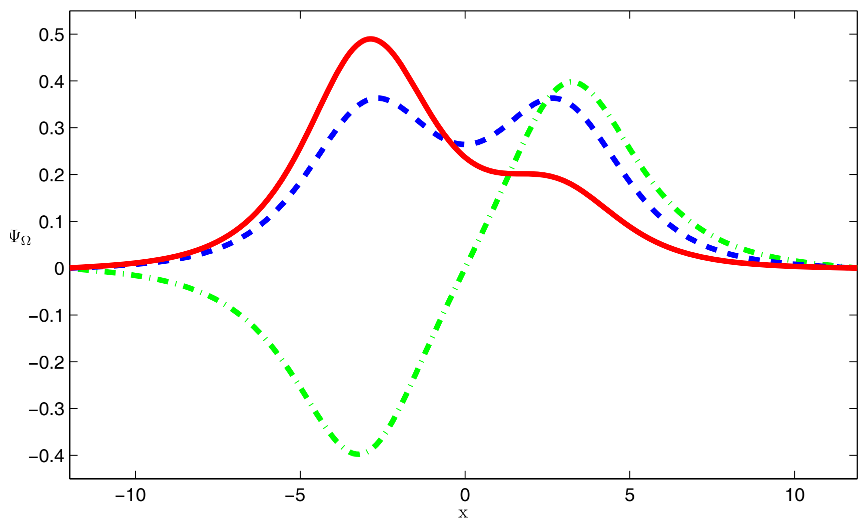
Bifurcation can occur only when L_+ or L_- are non-invertible. In particular this happens at $\phi \equiv 0$, $\Omega \in \sigma_{discrete}(-\Delta + V)$. We get bifurcation from a double e-value but one e-vector is related to rotational symmetry and can be "mode out."

$$F(\psi_\Omega, \Omega) = (-\Delta + V - \Omega)\psi_\Omega - |\psi_\Omega|^2\psi_\Omega = 0, \; \mathcal{N} = \|\psi_\Omega\|_{L^2}^2$$

Bifurcation diagram



Boundstates at $\mathcal{N} = 0.5$



How much can the branches be continued ?

$$F[\psi_\Omega, \Omega] = (-\Delta + V - \Omega)\psi_\Omega - |\psi_\Omega|^2\psi_\Omega = 0$$

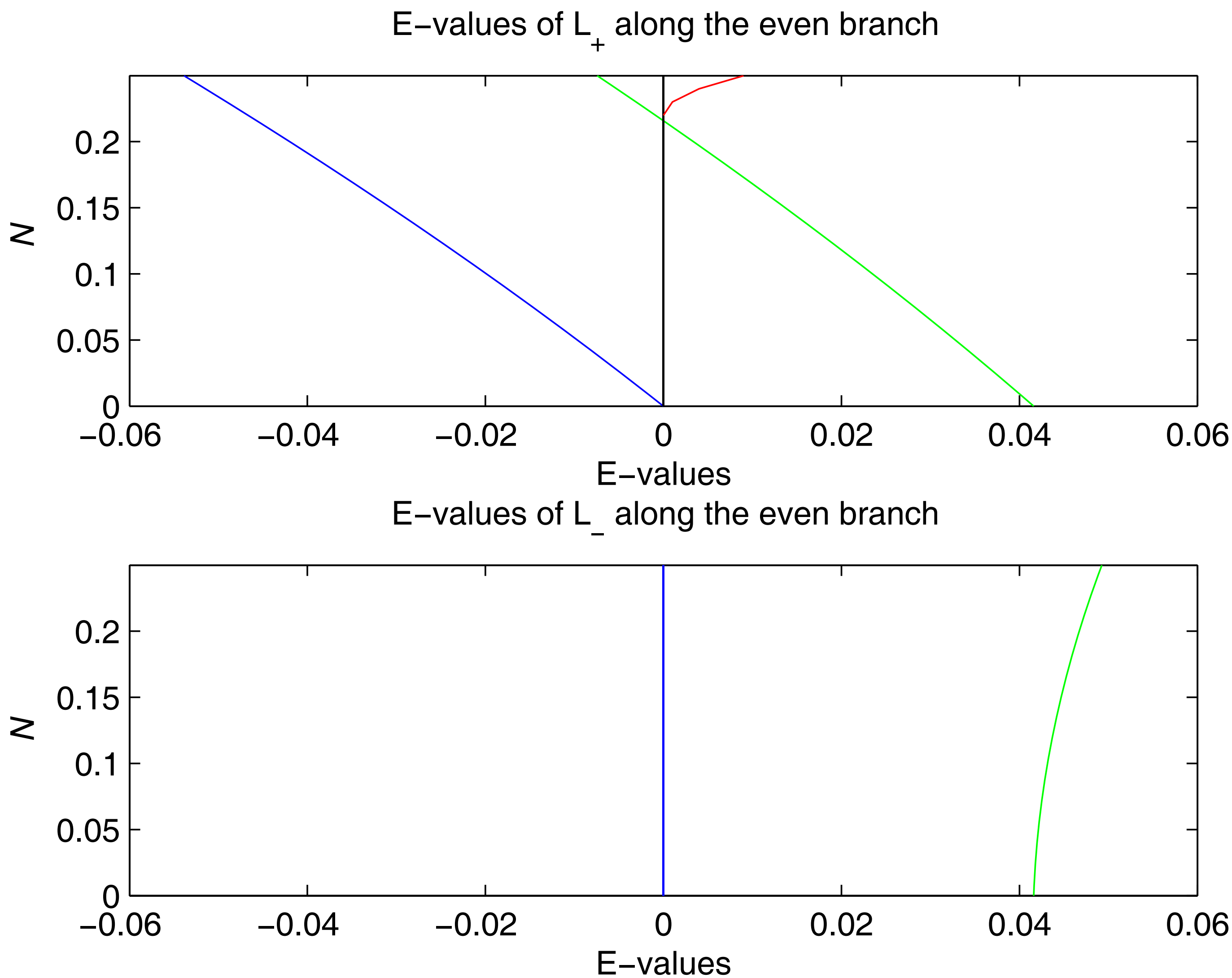
Fréchet derivative w.r.t first variable at a real valued solution $(\psi_\Omega,\ \Omega)$ in the direction $u + iv$:

$$DF_\psi(\phi, \Omega)[u + iv] = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

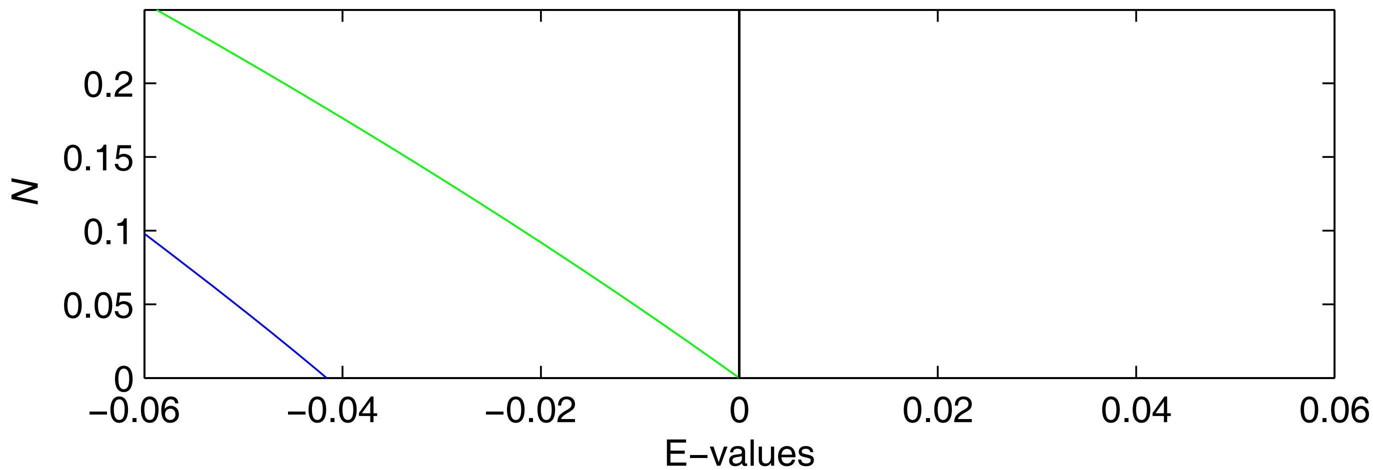
where

$$\begin{aligned} L_-[v] &= (-\Delta + V - \Omega)v - |\psi_\Omega|^2 v \\ L_+[u] &= (-\Delta + V - \Omega)u - 3|\psi_\Omega|^2 u \end{aligned}$$

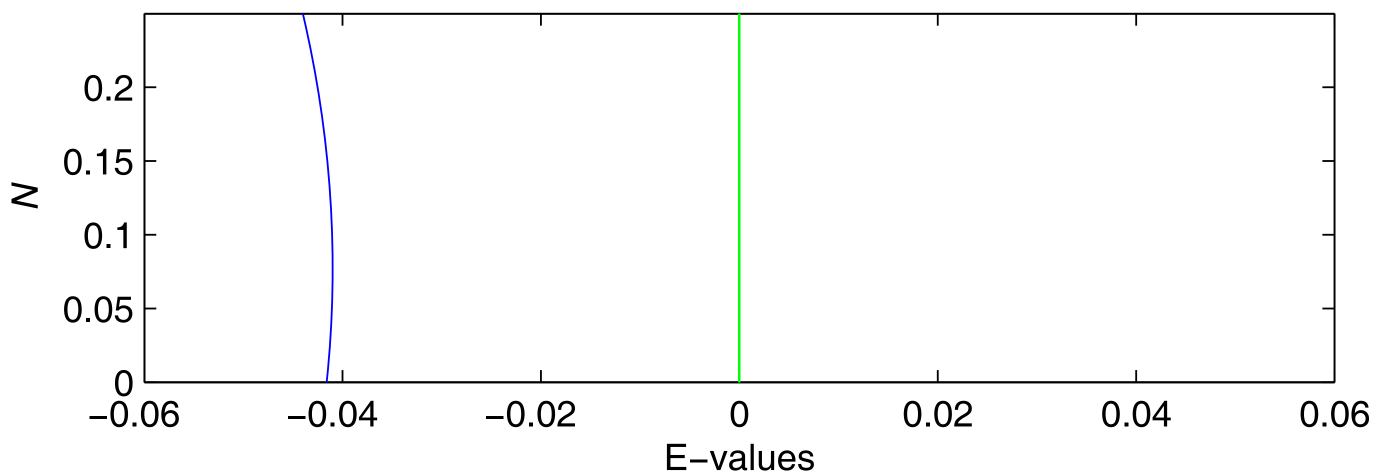
Note that $\psi_\Omega \in \ker L_-$ because of invariance of the set of solutions under rotations. Hence it does not lead to bifurcations. The $(\psi_\Omega,\ \Omega)$ branch can be continued as long as $\dim\ker L_+ = 0$ and $\dim\ker L_- = 1$. Also note that $L_+ < L_-$ for $\psi_\Omega \neq 0$.


E−values of L+ along the even branch
N
0.2
0.15
0.1
0.05
0
−0.06
−0.04
−0.02
0
0.02
0.04
0.06
E−values
E−values of L_ along the even branch
N
0.2
0.15
0.1
0.05
0
−0.06
−0.04
−0.02
0
0.02
0.04
0.06
E−values

E−values of L_+ along the odd branch



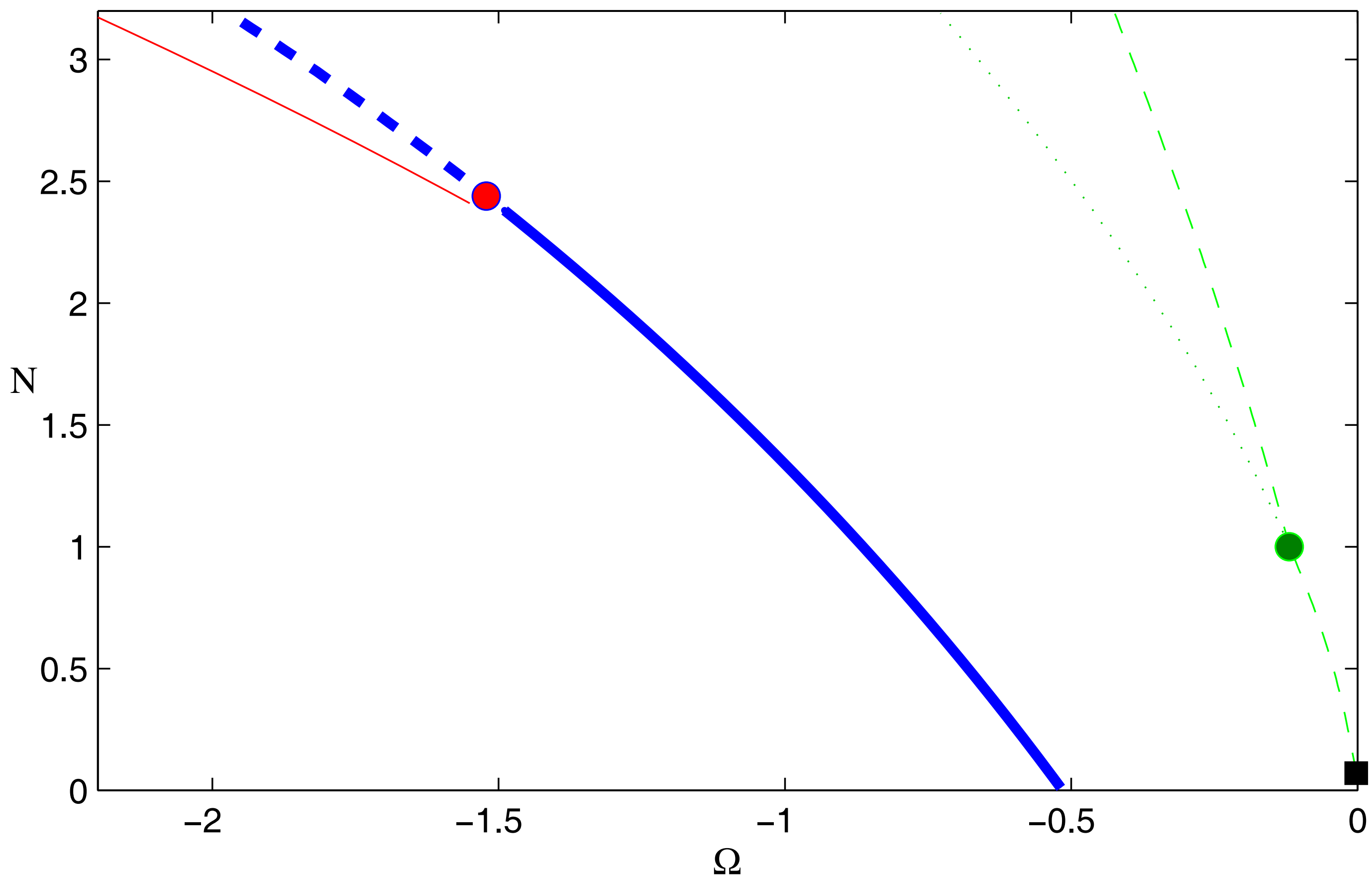
E−values of L_- along the odd branch



Conclusion: A bifurcation occurs along the branch corresponding to the lowest e-value of $(-\Delta + V)$ if and only if the second e-value of L_+ crosses zero. If this e-value is simple, exactly one other branch emerges, and it is transversal to the initial one.

No bifurcation can occur along the branch corresponding to the second lowest e-value of $(-\Delta + V)$ until the third e-value of L_+ crosses zero.

For repelling nonlinearities the situation is reversed.


3
2.5
2
N
1.5
1
0.5
0
−2
−1.5
−1
−0.5
0
Ω

Orbital Stability

The boundstate $\psi_\Omega(x)$ is stable if trajectories starting near its orbit $e^{i\Omega t}\psi_\Omega(x)$ remain close to its orbit (in Sobolev space H^1).

Linearized dynamics:

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

A degree type argument shows that if the number of negative e-values of L_+ and L_- differ by more than one, the boundstate is (linearly) unstable. This happens for the even branch past the bifurcation point.

A Lyapunov functional argument shows that if L_+ has at most one negative e-value, the stability is equivalent to $L_+ > 0$, $L_- > 0$ on $\{\psi_\Omega\}^\perp$.

These conditions are satisfied for the even branch at $\psi_\Omega = 0$. By ellipticity $L_- > 0$, and the conditions are violated only when an e-value of L_+ restricted to $\{\psi_\Omega\}^\perp$ crosses zero or $\frac{dN}{d\Omega} = 0$.

Hence, for $\Omega_1 - \Omega_0$ small, the even branch is stable until the bifurcation point when the asymmetric branch emerges and is stable. The sign of the e-values along the asymmetric branch can be established by linearizing the reduced system.

Comments

- We have a complete analysis of the bifurcation and stability of nonlinear boundstates near two relatively close linear boundstates. As opposed to previous results it does not rely on existence of global minimizers for energy (or on subcritical nonlinearities) but it does rely on perturbation theory. Extension beyond perturbation regimes is work in progress.

- The technique can be applied to more than two relatively close boundstates provided reductions to a manageable number of degrees of freedom can be found.

- More general nonlinearities can be treated but higher order ones lead to degenerate reduced systems.

Thank you!