Math 418, Final Exam, Part I
May 12, 2014

No calculators or other e-devices and no books or notes are allowed. Solve **ALL** problems. Time: 80 min.

[1] [10 pts.] Let \( m, n \in \mathbb{N} \) and \( d = \gcd(m, n) \in \mathbb{N} \). Show that
\[
\gcd(X^m - 1, X^n - 1) = X^d - 1 \quad \text{in } \mathbb{Z}[X].
\]

[2] [12 pts.] Let \( x_1, x_2, x_3 \) denote the roots of the polynomial \( X^3 + aX^2 + bX + c \).
(i) Express the symmetric polynomial \( p_5 = x_1^5 + x_2^5 + x_3^5 \) as a function in \( a, b, c \).
(ii) Do the same thing for the symmetric function
\[
D = (x_1 - x_2)^2(x_2 - x_3)^2(x_1 - x_3)^2.
\]

[3] [15 pts.] Suppose that \( F \) is a field and \( p(X) \in F[X] \) is a monic polynomial with \( \deg(p) > 0 \). Prove that the following three statements are equivalent:
(i) \( p(X) \) is irreducible in \( F[X] \).
(ii) The quotient ring \( F[X]/(p(X)) \) is an integral domain.
(iii) \( F[X]/(p(X)) \) is a field.

[4] [13=4+3+6 pts.] (i) Let \( R \) be a ring, \( M \) be a right \( R \)-module and \( N \) be a left \( R \)-module. What is the definition of the tensor product of \( M \) and \( N \) over \( R \) (viewed as a \( \mathbb{Z} \)-module)?
(ii) Sketch the construction of the tensor product.
(iii) Prove the following isomorphisms:
(iii.1) \( F \otimes_R M = 0 \) if \( F \) is a field, \( R \) a subring of \( F \) and \( M \) a torsion module over \( R \).
(iii.2) \( \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_d \) where \( m, n \in \mathbb{N} \), \( d = \gcd(m, n) \in \mathbb{N} \) and \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \).
(iii.3) \( F^m \otimes_F F^n \cong F^{mn} \) where \( F \) is a field.

Perfect score: 50 points.
Math 418, Final Exam, Part II
May 12, 2014

No calculators or other e-devices. Books and lecture notes are allowed. Solve ALL FIVE problems. Time: 80 min.

[1] [9 pts.] Let \( R = \mathbb{Z}[\sqrt{-n}] \) where \( n \) is a squarefree integer greater than 3. Prove that 2, \( \sqrt{-n} \), and \( 1 + \sqrt{-n} \) are irreducible in \( R \).

[2] [10 pts.] Consider the polynomial \( p(X) = X^3 - 6X^2 + 9X + 3 \).
   (i) Why is \( p(X) \) irreducible in \( \mathbb{Q}[X] \)?
   (ii) Consider the extension \( \mathbb{Q}(u) \) of \( \mathbb{Q} \) generated by a real root \( u \) of \( p(X) \). Express \( 3u^5 - u^4 + 2 \) in terms of the basis \( \{1, u, u^2\} \) of \( \mathbb{Q}(u) \).

[3] [9 pts.] (i) Show that in the field \( \mathbb{C} \) the subfields \( \mathbb{Q}(i) \) and \( \mathbb{Q}(\sqrt{2}) \) are isomorphic as \( \mathbb{Q} \)-vector spaces but not as fields.
   (ii) If \( n > 2 \) and \( \mu \) is a primitive \( n \)th root of unity, find \( [\mathbb{Q}(\mu + \mu^{-1}) : \mathbb{Q}] \).
   (iii) Which roots of unity are contained in the fields \( \mathbb{Q}(\sqrt{5}) \) and respectively \( \mathbb{Q}(\sqrt{-3}) \)?

[4] [10 pts.] (i) If \( F = \mathbb{Q}(\sqrt[12]{2}, \sqrt[12]{3}) \), find \( [F : \mathbb{Q}] \) and a basis of \( F \) over \( \mathbb{Q} \).
   (ii) Do the same for \( F = \mathbb{Q}(i, \sqrt[3]{3}, \omega) \), where \( i \in \mathbb{C}, i^2 = -1 \), and \( \omega \neq 1 \) is a cubic root of unity.

[5] [12 pts.] (i) Determine all possible rational canonical forms for a linear transformation with characteristic polynomial \( X^2(X^2 + 1)^2 \).
   (ii) Determine all possible Jordan canonical forms for a linear transformation with characteristic polynomial \( (X - 2)^3(X - 3)^2 \).

Perfect score: 50 points.