

Math 542, HW # 910

(due Wednesday, Dec 10)

- 1) Let D be a domain in \mathbb{C} and (f_n) be a sequence of uniformly bounded analytic functions on D . Let S be a subset of D such that S has a limit point in D , and (f_n) converges pointwise on S . Prove that (f_n) is uniformly Cauchy on compact subsets of D .

- 2) Let \mathfrak{F} be a set of analytic functions on \mathbb{D} . Show that \mathfrak{F} is relatively compact in $A(\mathbb{D})$ if and only if $\exists M_n \geq 0$ with $\limsup_n \sqrt[n]{M_n} \leq 1$ and $\forall f \in \mathfrak{F}, \forall N \geq 0$,

$$\left| \frac{f^{(n)}(0)}{n!} \right| \leq M_n.$$

- 3) Construct explicitly a meromorphic function on \mathbb{C} with poles exactly at $in, n = 1, 2, 3, \dots$, and corresponding principal part

$$\frac{n^{5/2}}{(z - in)^2}.$$

- 4) Let $D = \{z \in \mathbb{C} : 0 < \text{Im } z < \pi\}$ and let \mathfrak{F} denote the set of conformal mappings $f : D \rightarrow \mathbb{D}$ with $f\left(\frac{i\pi}{2}\right) = 0$.

(i) Construct explicitly a function $h \in \mathfrak{F}$.

(ii) Determine the set of all possible values of $f'\left(\frac{i\pi}{2}\right)$ for $f \in \mathfrak{F}$ and justify your answer.

- 5) Let $G = \left\{ z \in \mathbb{C} : \frac{\pi}{4} < \text{Arg} \frac{1+z}{1-z} < \frac{\pi}{2} \right\}$.

(i) Find the equation of the boundary of G in x, y coordinates.

(ii) Find a conformal mapping of G onto the open disk \mathbb{D} .

- 6) Suppose that F is analytic on a convex domain Ω .

(i) Suppose that $F'(z) \neq 0$ for all $z \in \Omega$. Give an example to show that F need not be one-to-one.

(ii) Suppose, in addition, that $\text{Re } F'(z) > 0$ for all $z \in \Omega$. Prove that F is one-to-one.

- 7) Show that every conformal map of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ is of the form

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1.$$

- 8) (i) Find explicitly a conformal mapping of the horizontal strip $\{z \in \mathbb{C} : 0 < \text{Im } z < 1\}$ onto \mathbb{D} .

- (ii) Find a conformal mapping of $\{z \in \mathbb{D} : \operatorname{Im} z > 0\}$ onto \mathbb{D} .
- 9) (i) Find explicitly a conformal mapping from $\mathbb{D} \setminus [0, 1)$ onto \mathbb{D} .
(ii) Find an analytic mapping from $\mathbb{D} \setminus [1/2, 1)$ onto \mathbb{D} .
- 10) (i) Find a conformal mapping f from
- $$\Omega = \{z \in \mathbb{C} : |z + i| < \sqrt{2}, |z - i| < \sqrt{2}\}$$
- onto \mathbb{D} such that $f(0) = 0$.
- (ii) Is there a conformal mapping $g : \Omega \rightarrow \mathbb{D}$ such that $g(0) = 0$ and $g(1/2) = 9/10$? Justify your answer.