Math 231, Sections AL1 and CL1

Homework 3 (some solutions)

1) Compute $I = \displaystyle\int \frac{x-1}{x^2+2x+5}\,dx$ where x is in $\mathbb{R}$.

Completing the square we write $x^2+2x+5 = (x+1)^2+4$ and

$$I = \int \frac{x-1}{x^2+2x+5}\,dx = \int \frac{(x+1)-2}{(x+1)^2+4}\,dx.$$

Taking $u = x+1$ this gives

$$I = \int \frac{u-2}{u^2+4}\,du = \int \frac{u}{u^2+4} - 4\int \frac{du}{u^2+2^2} = \frac{1}{2}\ln(u^2+4) - 4\frac{1}{2}\arctan\frac{u}{2} + C$$
$$= \frac{1}{2}\ln(x^2+2x+5) - 2\arctan\frac{x+1}{2} + C.$$

2) Compute $I = \displaystyle\int \frac{(a-b)x}{x^2-(a+b)x+ab}\,dx$ where $a < x < b$.

The decomposition $x^2-(a+b)x+ab = (x-a)(x-b)$ and $a \neq 0$ yield the simple fraction decomposition

$$\frac{(a-b)x}{x^2-(a+b)x+ab} \equiv \frac{A}{x-a} + \frac{B}{x-b} \qquad \Big|\cdot (x-a)(x-b)$$

or equivalently

$$(a-b)x \equiv A(x-b) + B(x-a). \tag{1}$$

Plugging $x = a$, respectively $x = b$, in (1) we get $A = a$ and respectively $B = -b$, so that

$$I = \int \left(\frac{a}{x-a} - \frac{b}{x-b}\right)dx = a\ln|x-a| - b\ln|x-b| = a\ln(x-a) - b\ln(b-x).$$

3) Compute $I = \displaystyle\int \frac{x^4+1}{(x-1)^2}\,dx$ where $x > 1$.

Long division gives

$$x^4+1 = (x^2-2x+1)(x^2+2x+3) + 4x - 2 = (x-1)^2(x^2+2x+3) + 4x - 2,$$

hence

$$\frac{x^4+1}{(x-1)^2} = x^2+2x+3+\frac{4x-2}{(x-1)^2} = x^2+2x+3+\frac{4(x-1)+2}{(x-1)^2} = x^2+2x+3+\frac{4}{x-1}+\frac{2}{(x-1)^2}$$

and therefore

$$I = \frac{x^3}{3} + x^2 + 3x + 4\ln(x-1) - \frac{2}{x-1}.$$

An alternative way is to make from the beginning the substitution $u = x - 1$, getting

$$\begin{aligned} I &= \int \frac{(u+1)^4+1}{u^2}\,du = \int \frac{u^4+4u^3+6u^2+4u+2}{u^2}\,du \\ &= \int \left(u^2+4u+6+\frac{4}{u}+\frac{2}{u^2}\right)du = \frac{u^3}{3}+2u^2+6u+4\ln|u|-\frac{2}{u}+C \\ &= \frac{(x-1)^3}{3}+2(x-1)^2+6(x-1)+4\ln(x-1)-\frac{2}{x-1}+C. \end{aligned}$$

4) Compute $I = \displaystyle\int \frac{x^2-2}{(x^2+1)(x-2)^2}\,dx$ where $x > 2$.

The algorithm of decomposition into simple fractions gives

$$\frac{x^2-2}{(x^2+1)(x-2)^2} \equiv \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+1} \quad \Big| \cdot (x-2)^2(x^2+1),$$

or equivalently

$$\begin{aligned} x^2-2 &\equiv A(x-2)(x^2+1)+B(x^2+1)+(Cx+D)(x^2-4x+4) \\ &= x^3(\underbrace{A+C}_{=0})+x^2(\underbrace{-2A+B-4C+D}_{=1})+x(\underbrace{A+4C-4B}_{=0})\underbrace{-2A+B+4D}_{=-2}. \end{aligned} \tag{2}$$

Plugging $x = 2$ in (2) we find $B = \frac{2}{5}$. Identifying the coefficients of x then gives $A = -\frac{8}{25}$, $C = \frac{8}{25}$, $D = \frac{6}{25}$, therefore

$$\begin{aligned} I &= A\ln(x-2)-\frac{B}{x-2}+\frac{C}{2}\ln(x^2+1)+D\arctan x+\mathcal{C} \\ &= -\frac{8}{25}\ln(x-2)-\frac{2}{5}\cdot\frac{1}{x-2}+\frac{4}{25}\ln(x^2+1)+\frac{6}{25}\arctan x+\mathcal{C}. \end{aligned}$$

5) Compute $I = \displaystyle\int \frac{x^2+2}{x(x^2+1)(x^2+4)}\,dx$ where $x < 0$.

Since x^2+1 and x^2+4 are irreducible polynomials we can write

$$\frac{x^2+2}{x(x^2+1)(x^2+4)} \equiv \frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2+4} \quad \Big| \cdot x(x^2+1)(x^2+4),$$

or equivalently

$$\begin{aligned} x^2+2 &\equiv A(x^2+1)(x^2+4)+(Bx+C)((x^3+4x)+(Dx+E)(x^3+x) \\ &= x^4(\underbrace{A+B+D}_{=0})+x^3(\underbrace{C+E}_{=0})+x^2(\underbrace{5A+4B+D}_{=1})+x(\underbrace{4C+E}_{=0})+\underbrace{4A}_{=2}. \end{aligned} \tag{3}$$

Identifying the coefficients of x in (3) we find

$$A = \frac{1}{2}, \quad B = -\frac{1}{3}, \quad C = 0, \quad D = -\frac{1}{6}, \quad E = 0.$$

Therefore

$$\int \frac{x^2+2}{x(x^2+1)(x^2+4)}\,dx = A\int \frac{dx}{x} + B\int \frac{x}{x^2+1}\,dx + D\int \frac{x}{x^2+4}\,dx$$
$$= \frac{1}{2}\ln(-x) - \frac{1}{6}\ln(x^2+1) - \frac{1}{12}\ln(x^2+4) + \mathcal{C}.$$

6) Compute $I = \displaystyle\int \frac{x^2}{(x+1)^2(x^2-6x+10)}\,dx$ where $x > -1$.

The polynomial $x^2 - 6x + 10 = (x-3)^2 + 1$ is irreducible, hence

$$\frac{x^2}{(x+1)^2(x^2-6x+10)} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-6x+10} \quad \Bigg| \cdot (x+1)^2(x^2-6x+10),$$

or equivalently

$$\begin{aligned} x^2 &\equiv A(x+1)(x^2-6x+10) + B(x^2-6x+10) + (Cx+D)(x+1)^2 \\ &= x^3(\underbrace{A+C}_{=0}) + x^2(\underbrace{-5A+B+2C+D}_{=1}) + x(\underbrace{4A-6B+C+2D}_{=0}) \\ &+ \underbrace{10A+10B+D}_{=0}. \end{aligned} \tag{4}$$

Plugging $x=-1$ in (4) we find $B = \frac{1}{17}$. Further identification of the coefficients of x in (4) leads to $A = -\frac{26}{289}$, $C = \frac{26}{289}$, $D = \frac{90}{289}$.

We finally get

$$\begin{aligned} I &= A\int \frac{dx}{x+1} + B\int \frac{dx}{(x+1)^2} + \int \frac{Cx+D}{(x-3)^2+1}\,dx \\ (u = x-3) \quad &= A\ln(x+1) - \frac{B}{x+1} + \int \frac{C(u+3)+D}{u^2+1}\,du \\ &= A\ln(x+1) - \frac{B}{x+1} + \frac{C}{2}\ln(u^2+1) + (3C+D)\arctan u + \mathcal{C} \\ &= -\frac{26}{289}\ln(x+1) - \frac{1}{17(x+1)} + \frac{13}{289}\ln(x^2-6x+10) + \frac{168}{289}\arctan(x-3) + \mathcal{C}. \end{aligned}$$