

# Math 542, HW # 6

(due Wed, Oct 29)

1) Suppose that  $f$  is an entire function and that there are positive constants  $A$  and  $m$  with  $|f(z)| \leq A|z|^m$  if  $|z| \geq R_0$ . Show that  $f$  is a polynomial of degree at most  $m$ .

2) Find the Laurent expansions of the following rational functions in powers of  $z$  and  $\frac{1}{z}$  in the indicated regions.

(a)  $f(z) = \frac{z+2}{z^2-z-2}$  in  $1 < |z| < 2$  and then in  $2 < |z| < \infty$ .

(b)  $f(z) = \frac{z^2+2z-4}{(z^2-9)(z+1)}$  in  $|z| < 1$  and then in  $1 < |z| < 3$ .

3) Evaluate

$$I = \int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx.$$

4) Evaluate

$$I = \int_0^{\infty} \frac{x^\lambda}{x^2 + 2x \cos \omega + 1} dx \quad (-\pi < \omega < \pi, 0 < \lambda < 1).$$

5) Evaluate by contour integration

$$I_n = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n} \quad (n \in \mathbb{N}).$$

6) Use the argument principle to determine the number of zeros of  $f$  in the first quadrant.

(a)  $f(z) = z^9 + 5z^2 + 3$ .

(b)  $f(z) = z^7 + 6z^3 + 7$ .

7) Prove that if  $0 < |a| < 1$  and  $n \in \mathbb{N}$ , then the equation  $(z-1)^n e^z = a$  has precisely  $n$  roots, each with multiplicity one, in the half-plane  $\operatorname{Re} z > 0$ .

8) Without using the Fundamental Theorem of Algebra, prove for any polynomial  $P$  of degree  $\geq 1$  that

$$\lim_{R \rightarrow \infty} \oint_{|z|=R} \frac{P'(z)}{P(z)} dz = 2\pi i n.$$

Deduce the Fundamental Theorem of Algebra from this equality.

9) Let  $f : \mathbb{D} \rightarrow \{z : \operatorname{Re} z > 0\}$  be an analytic function. Show that

$$\frac{1 - |z|}{1 + |z|} |f(0)| \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|} |f(0)|, \quad z \in \mathbb{D},$$

and

$$|f'(0)| \leq 2|\operatorname{Re} f(0)|.$$

10) Suppose  $f : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  is continuous and analytic on  $\mathbb{D}$ . Assume that  $f$  has zeros at  $z_1, \dots, z_n$  of orders  $k_1, \dots, k_n$  respectively. Show that

$$|f(z)| \leq \prod_{j=1}^n \left| \frac{z - z_j}{1 - \overline{z_j} z} \right|^{k_j}, \quad z \in \mathbb{D}.$$

Suppose equality holds for some  $z \in \mathbb{D}$  with  $z \neq z_j$ ,  $j = 1, \dots, n$ . Find a formula for  $f(z)$ .