Math 542, HW # 7

(due Mon, Nov 10)

1) Suppose that $u : \mathbb{D} \to [0,\infty)$ is a harmonic function with $u(0) = 1$. How large can $u\left(\frac{1}{2}\right)$ be ? How small can it be ? Give the best possible bounds and justify your answers.

2) Suppose that $f : \mathbb{D} \to \mathbb{D}$ is analytic and f has a zero of multiplicity two at $z = \frac{1}{2}$. Obtain the best upper bound you can for $\left|f\left(\frac{3}{4}\right)\right|$ and justify your answer.

3) Suppose that Ω is a domain and $f : \Omega \to \mathbb{C}^*$ is continuous.
   (i) If f^2 is analytic on Ω, show that f is analytic on Ω.
   (ii) Is the same property true for $f : \Omega \to \mathbb{C}$?

4) Given Ω be a domain in $\mathbb{C}$ and $K \subseteq \Omega$ a compact, show that there exists a constant $C > 0$ such that for every harmonic function $h : \Omega \to (0,\infty)$,

$$\frac{\sup_K h}{\inf_K h} \leqslant C.$$

5) Show that there exists a constant $C > 0$ such that for every harmonic function $u : \mathbb{D} \to [-1,1]$

$$\sup_{|z| \leqslant \frac{1}{2}} \left|\frac{\partial u}{\partial x}\right| \leqslant C.$$

6) Every function u, harmonic on a simply connected domain Ω, can be represented in the form

$$u(x,y) = \log|f(z)|,$$

where $f(z)$ is analytic and non-vanishing on Ω. Is the result true when Ω is an annulus ?

7) Suppose that u is a harmonic function on the domain Ω. Show, for all positive integers $k,\ell \geqslant 1$, that

$$\frac{\partial^{k+\ell} u(x,y)}{\partial x^k \partial y^\ell}$$

is harmonic on Ω.

8) Consider the analytic function $f : B_R(0) \to \mathbb{C}$ and

$$I(r) := \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt.$$

   (i) Prove that $I : [0,R) \to [0,\infty)$ is continuous and strictly increasing when f is not constant.
   (ii) $|f(0)|^2 \leqslant I(r), \qquad \forall r \in [0,R)$.

(iii) Prove that if (f_n) is a sequence of analytic functions that converges in L^2 on each compact subset of $B_R(0)$, then (f_n) converges uniformly on each compact subset of $B_R(0)$.

9) Prove that for each smooth and compactly supported function g on $\mathbb{C}$, the function f defined by

$$f(w) = \frac{1}{2\pi} \iint_{\mathbb{C}} g(z) \log|z-w| \, dx dy, \qquad w \in \mathbb{C},$$

is a solution of $\Delta f = g$.

10) (i) Let $0 \leqslant r_1 < r_2$. For each harmonic function u on $A(r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z| < z_2\}$, there exist constants A and B such that

$$\int_0^{2\pi} u(re^{it}) \, dt = A \log r + B.$$

(ii) Prove that if u is harmonic on $\mathbb{D} \setminus \{0\}$ and continuous on $\mathbb{D}$, then u is harmonic on $\mathbb{D}$.