Math 542, HW # 8

(due Friday, Nov 21)

1) Show that for each real $\lambda > 1$, the equation $z + e^{-z} = \lambda$ has exactly one root z_0 with $\operatorname{Re} z_0 > 0$.

2) Let $f(z)$ be a nonconstant meromorphic function in the plane such that
$$f(z+1) = f(z) = f(z+i), \qquad \forall z \in \mathbb{C}.$$
Prove that f has the same number of zeros and poles in the half-open rectangle
$$R = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Re} z < 1, 0 \leqslant \operatorname{Im} z < 1\}.$$

3) Suppose f is analytic and bounded on the unit disk $\mathbb{D}$ and $f(0) \neq 0$. Let $\{s_n\}$ be the zeros of f (counted with multiplicities). Show that
$$\sum_n \big(1 - |s_n|\big) \; < \; \infty.$$
(Hint: Use Jensen's formula on $|z| < r < 1$.)

4) Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function such that $f(0) = 0$ and f is not a rotation of the form $f(z) = cz$ ($|c| = 1$). Let $f_n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$. Prove that f_n converges to zero uniformly on compact subsets of $\mathbb{D}$.

5) Show
$$\Gamma\left(\frac{1}{6}\right) \; = \; 2^{-\frac{1}{3}} \sqrt{\frac{3}{\pi}}\, \Gamma\left(\frac{1}{3}\right)^2.$$

6) Show ($y \in \mathbb{R}$, $y \neq 0$)
$$|\Gamma(iy)|^2 \; = \; \frac{\pi}{y \sinh(\pi y)}, \qquad \left|\Gamma\left(\frac{1}{2} + iy\right)\right|^2 \; = \; \frac{\pi}{\cosh(\pi y)}.$$

7) (Gauss Multiplication Formula) For any $n \in \mathbb{N}$
$$(2\pi)^{\frac{n-1}{2}} \Gamma(z) \; = \; n^{z - \frac{1}{2}}\, \Gamma\left(\frac{z}{n}\right) \Gamma\left(\frac{z+1}{n}\right) \cdots \Gamma\left(\frac{z+n-1}{n}\right).$$

8) Show
$$\cos(\pi z) \; = \; \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right) \; = \; \prod_{n=-\infty}^{\infty} \left(1 - \frac{2z}{2n-1}\right) e^{\frac{2z}{2n-1}}.$$

9) (i) Find an entire function f whose zero set is exactly

$$\{\ln 1, \ln 2, \ln 2, \ln 3, \ln 3, \ln 3, \ln 4, \ln 4, \ln 4, \ln 4, \cdots\}.$$

(ii) Find a meromorphic function on $\mathbb{C}$ with a pole at in and principal part $\frac{\sqrt{n}}{z-in}$ for each $n \in \mathbb{N}$.

Provide the details of your reasoning for both questions.

10) Let $\{r_n\}_{n\geqslant 2}$ be an enumeration of $\mathbb{Q} \cap [0, 2\pi)$. Show that

$$f(z) = \prod_{n=2}^{\infty} e^{-ir_n} \frac{\left(1-\frac{1}{n^2}\right)e^{ir_n} - z}{1-\left(1-\frac{1}{n^2}\right)e^{-ir_n} z}$$

defines a bounded analytic function on $\mathbb{D}$. Can f be extended analytically to any connected set that is larger than $\mathbb{D}$? Justify your answer.