

Math 542, HW # 9

(due Wednesday, Dec 3)

- 1) Prove that the function

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad |z| < 1,$$

does not extend analytically to any open set strictly larger than the (closed) unit disk.

- 2) Let g be a continuous function on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Prove that the following statements are equivalent:

- (i) g is uniformly approximable on \mathbb{T} by polynomials in z .
- (ii) g has a continuous extension to the closed unit disk $\overline{\mathbb{D}}$ which is analytic on \mathbb{D} .

- 3) Suppose that (a_n) and (b_n) are disjoint sequences of complex numbers such that $\sum_n |a_n - b_n| < \infty$ and $|a_n| \rightarrow \infty$. Show that

$$\prod_{n=1}^{\infty} \frac{z - a_n}{z - b_n}$$

defines a meromorphic function on \mathbb{C} .

- 4) Display a meromorphic function on \mathbb{C} having simple poles exactly in $S = \{\sqrt{n} : n \in \mathbb{N}\}$ with corresponding residues $\text{Res}(f; \sqrt{n}) = \sqrt{n}$.

- 5) (i) Suppose that $f \in A(D)$ and $\overline{B_R(a)} \subseteq D$. Prove that

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{it})|^2 r dr dt.$$

- (ii) Which of the following two families is normal:

$$\mathfrak{F}_1 = \left\{ f \in A(D) : \iint_D |f(x + iy)|^2 dx dy \leq 1 \right\};$$
$$\mathfrak{F}_\infty = \left\{ f \in A(D) : \iint_D |f(x + iy)|^2 dx dy < \infty \right\}.$$

- 6) (i) Let $z_0 \in \mathbb{D}$. Prove that

$$\mathfrak{F} = \{f : \mathbb{D} \rightarrow \mathbb{C} : f \text{ analytic, } f(z_0) = 1, \text{Re } f(z) > 0 \text{ for all } z \in \mathbb{D}\}$$

is a normal family.

- (ii) Let

$$\mathcal{H} = \{u : \mathbb{D} \rightarrow (0, \infty) : u \text{ harmonic, } u(z_0) = 1\}.$$

Show that

$$\forall w \in \mathbb{D}, \exists 0 < M_w < \infty, \sup_{u \in \mathcal{H}} \left| \frac{\partial u}{\partial x}(w) \right| \leq M_w.$$

(iii) Use a normal family argument to prove that there exists $0 < C < \infty$ such that

$$h\left(\frac{1}{2}\right) \leq Ch(0)$$

for every harmonic function $h : \mathbb{D} \rightarrow (0, \infty)$.

(iv) Is the statement in (ii) still true when \mathbb{D} is replaced by a domain which is not simply connected?

7) Let $M > 0$. Show that

$$\mathfrak{F} = \left\{ f \in A(\mathbb{D}) : \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})| dt \leq M \right\}$$

is a normal family.

8) Let $D = \{z = x + iy : x > y > 0\}$ and \mathfrak{F} the collection of analytic functions $f : D \rightarrow \mathbb{D}$.

(i) Is \mathfrak{F} a normal family?

(ii) Give an example of a non-constant function in \mathfrak{F} .

(iii) Prove that

$$M := \sup_{f \in \mathfrak{F}} \left| f''\left(1 + \frac{i}{2}\right) \right| < \infty.$$

(iv) Does there exist $f \in \mathfrak{F}$ such that $|f''(1 + \frac{i}{2})| = M$?

9) Let $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and

$$\mathfrak{F} = \{f : H \rightarrow H : f \text{ analytic, } f(1) = 1\}.$$

Show that

$$\exists g \in \mathfrak{F}, \quad |g'''(4)| = \sup_{f \in \mathfrak{F}} |f'''(4)|.$$

10) If the family \mathfrak{F} of analytic functions in a region D is not normal, show that there exists $z_0 \in D$ such that \mathfrak{F} is not normal in any neighborhood of z_0 . (Hint: Use a compactness argument.)