Spectral Theory of Dynamical Systems

We saw how to phrase ergodicity in terms of the operator $Tf = \int f d\mu$ on the Hilbert space $L^2(X, \mathcal{B}, \mu)$. Let us see that $\{T^n f\}_{n \geq 0}$ is a weakly stationary sequence in $L^2$ and how it can be related to a measure on $\Omega = \mathbb{R}/\mathbb{Z}$ (the spectral measure).

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A sequence of complex numbers $\{a_k\}_{k \in \mathbb{Z}}$ is nonnegative definite if $a_{-k} = \overline{a_k}$ and for all $n$, $\forall \{p_i, \ldots, p_n \} \in \mathcal{I}$ we have

$$\sum_{i,j \in \mathbb{N}} p_i \overline{p_j} a_{i-j} \geq 0.$$ 

Lemma 1 (Bochner-Herglotz)

Let $\gamma \colon \mathbb{Z} \to \mathbb{R}$ s.t. $\{\gamma(k)\}_{k \in \mathbb{Z}}$ is nonnegative definite. Then there exists a nonnegative bounded measure $\nu$ on $\Omega$ such that for all $k \in \mathbb{Z}$

$$\gamma(k) = \int_{\Omega} e^{2\pi ik \cdot t} d\nu(t).$$
Proof

By assumption

\[
\sum_{n,m=1}^{N} e^{-2\pi i (n-m)x} \gamma(n-m) > 0.
\]

Let us evaluate this sum as:

\[
\sum_{n,m=1}^{N} e^{-2\pi i (n-m)x} \gamma(n-m) = \sum_{n=1}^{N} \sum_{p=n-N}^{n-1} e^{-2\pi i px} \delta(p) =
\]

\[
= \sum_{p=-N+1}^{N-1} e^{-2\pi i px} \delta(p) \sum_{1 \leq n \leq N \atop p+1 \leq n \leq p+N} 1 =
\]

\[
= \sum_{p=-N+1}^{N-1} e^{-2\pi i px} \delta(p) \left[ \min(N, p+N) - \max(1, p+1) \right]
\]

\[
= \sum_{p=-N+1}^{N-1} e^{-2\pi i px} \delta(p) \left( N - 1 \right)
\]

Let \( \delta_N(p) = \delta(p). \frac{1}{1 - \frac{|p|}{N}} \) on \([-N+1, N-1]\)

\[
\begin{align*}
\hat{\gamma}_N(x) &= \sum_{p \in \mathbb{Z}} e^{-2\pi i px} \delta_N(p) \\
\hat{\delta}_N(-x) &= \hat{\gamma}_N(x). \text{ Since } \delta_N \text{ has compact support,}
\end{align*}
\]
Then $g_N$ is bounded and continuous.

Moreover,

\[
\int \frac{g_N(x)}{2\pi i} e^{2\pi i x r} \, dx = \sum_{p \in \mathbb{Z}} \delta_N(p) \int \frac{e^{2\pi i x (r-p)}}{2\pi i} \, dx = \delta_N(r)
\]

and, in particular, $\delta_N(0) = \int g_N(x) \, dx < \infty$.

The family of measures $\{\nu_N\}_{N \in \mathbb{N}}$,

where $d\nu_N(x) = g_N(x) \, dx$,

is weakly compact.

$\Rightarrow$ a subsequence $\{N_e\}_{e}$ st.

\[
\nu_{N_e} \xrightarrow{e \to \infty} \nu \text{ weakly, where } \nu \text{ is some nonnegative measure on } \mathbb{T}.
\]

We have:
\[
\lim_{\ell \to \infty} \varphi_{N_\ell}(r) = \oint \frac{e^{2\pi i r \cdot \nu}}{\Pi} d\nu(x).
\]

Since \( \lim_{N \to \infty} \varphi_N(r) = \varphi(r) \), we get that \( \forall r \in \mathbb{R} \)

\( \varphi(r) = \oint \frac{e^{2\pi i r \cdot \nu}}{\Pi} d\nu(x) \), i.e.

\( \hat{\varphi}(k) = \varphi(k) \)

Bochner–Herglotz lemma represents a nonnegative definite sequence as the Fourier transform of a measure. Another useful representation is due to Ky Fan:

**Theorem 2** (Ky Fan, 1946 Annals of Math.)

\( \{x_k\}_{k \in \mathbb{Z}} \) is nonnegative definite if and only if there exists a sequence of vectors \( \{X_k\}_{k \in \mathbb{Z}} \) in a Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) such that

\( \langle X_h, X_k \rangle = \alpha_{h-k} \)
Define a sequence \( \{X_k\}_{k \in \mathbb{N}} \subset H \) such that \( \langle X_h, X_k \rangle \) depends only on \( h-k \), called stationary in weak sense.

Consider a measure preserving transformation \( T \) on \((X, \mathcal{B}, \mu)\). The operator \( U_T \) on \( H = L^2(X, \mathcal{B}, \mu) \) given by \( U_T f = f \circ T \) is an isometry. Let \( f \in H \). Then the sequence \( \{U_T^k f\}_{k \geq 0} \) is weakly stationary.

\[
\langle U_T^k f, U_T^h f \rangle = \begin{cases} 
\langle U_T^{h-k} f, f \rangle = \mathbb{C}_{h-k} & \text{if } h-k \\
\langle f, U_T^{k-h} f \rangle = \mathbb{C}_{h-k} & \text{if } h < k
\end{cases}
\]
In general we have the following:

**Lemma 3.** Let $T$ be a contraction in a Hilbert space $H$. Define $T_n = \begin{cases} T^n & , n \geq 0 \\ (T^*)^{-n} & , n < 0 \end{cases}$

Let $f \in H$.

Then the sequence $\{\langle T_n f, f \rangle \}_{n \in \mathbb{Z}}$ is nonnegative definite, and $\exists \mu$ nonnegative bounded measure $\sqrt{\mu}$ on $\mathbb{T}$ (called the spectral measure of $T$ at $f$) such that

$$\langle T_n f, f \rangle = \int_{\mathbb{T}} e^{2\pi i nx} \sqrt{\mu}(x) \, d\mu(x) \quad \forall n \in \mathbb{Z}.$$ 

**Proof.** For isometries $T$ this follows from Kyton's Theorem and Bochner - Helgolatz's lemma.

For contractions the proof is simple (see Lemma 1.2.1 in [M. Weber, "Dynamical Systems and Processes"]). \qed
Proposition 4 (Spectral Inequality)

Let $T$ be a contraction in a Hilbert space $H$, and let $p(x)$ be a polynomial. Then $\forall f \in H$

$$||p(T)f||^2 \leq \int_T |p(e^{2\pi i x})|^2 \phi_f(x)$$

where $\phi_f$ is the spectral measure of $T$ at $f$.

Proof

By induction on the degree of $p$. If $\deg p = 0$, then the result is trivial. Let us assume that the result is proven for polynomials of degree $k-1$. Let $p(y) = a_0 + a_1 y + a_2 y^2 + \ldots + a_k y^k$.

Consider the polynomials

$q(y) = p(y) - a_0 = a_1 y + a_2 y^2 + \ldots + a_k y^k$ and

$$u(y) = \frac{q(y)}{y} = a_1 + a_2 y + a_3 y^2 + \ldots + a_k y^{k-1}.$$

We have $|p(y)|^2 = |a_0 + q(y)|^2 = |a_0|^2 + 2a_0 q(y) + |q(y)|^2$ and
\[ \| p(T)f \| \leq \| (\alpha_0 I + q(T))f \| \leq \]
\[ = |\alpha_0|^2 \| f \|^2 + \langle \alpha_0 f, q(T)f \rangle + \]
\[ + \langle q(T)f, \alpha_0 f \rangle + \| q(T)f \|^2 \]

**Induction hypothesis:**
\[ \| u(T)f \|^2 \leq \int_{\mathbb{T}} |u(e^{2\pi i x})|^2 d\nu_f(x). \]

Since \( T \) is a contraction, then
\[ \| q(T)f \| = \| T \cdot u(T)f \| \leq \| u(T)f \| \]

Moreover, since \( |u(e^{2\pi i x})| = |q(e^{2\pi i x})| \), we get
\[ \| q(T)f \|^2 \leq \int_{\mathbb{T}} |q(e^{2\pi i x})|^2 d\nu_f(x). \]

Next,
\[ \langle \alpha_0 f, q(T)f \rangle = \int_{\mathbb{T}} \alpha_0 \overline{q(e^{2\pi i x})} d\nu_f(x) \quad \text{and} \]
\[ \langle q(T)f, \alpha_0 f \rangle = \int_{\mathbb{T}} q(e^{2\pi i x}) \overline{\alpha_0} d\nu_f(x). \]

Putting everything together,
\[ \| p(T)f \|^2 \leq \int_{\mathbb{T}} \left( |\alpha_0|^2 + \alpha_0 q(e^{2\pi i x}) + \overline{q(e^{2\pi i x})} \alpha_0 + |q(e^{2\pi i x})|^2 \right) d\nu_f(x) \]
\[ = \int_{\mathbb{T}} |p(e^{2\pi i x})|^2 d\nu_f(x). \]
Here is an application of the spectral inequality: Let

\[ A_n^T = \frac{S_n^T}{n} \]

And

\[ S_n f = f + T f + T^2 f + \cdots + T^{n-1} f \]

where \( T \) is a contraction in the Hilbert space \( H \). Then

\[ \| A_n f - A_m f \| \leq \int_{-\pi}^{\pi} |V_n(\theta) - V_m(\theta)|^2 d\mu f(\theta) \]

where \( \mu f \) is the spectral measure of \( T \) at \( f \), and

\[ V_k(\theta) = \frac{1}{\ell} \sum_{k=0}^{\ell-1} e^{ik\theta} \ . \]

We have that

\[ \lim_{n,m \to \infty} |V_n(\theta) - V_m(\theta)| = 0 \quad \forall \theta \]

Moreover

\[ |V_k(\theta)| \leq 1 \ . \]

Dominated convergence \( \Rightarrow \lim_{n,m \to \infty} \int_{-\pi}^{\pi} |V_n(\theta) - V_m(\theta)|^2 d\mu f(\theta) = 0 \ . \]

That is:

\[ \| A_n f - A_m f \|^2 \to 0, \quad \text{i.e.} \quad A_n f \text{ is a Cauchy sequence in } H \ . \]

\( \Rightarrow \) \( A_n f \) converges in \( H \). (We retrieved the convergence part of von Neumann's theorem.)
We saw that sequences of the form \( \{ T^n \} \) in the Hilbert space \( L^2(\mathbb{X}, \mathbb{B}, \mu) \) are stationary in weak sense. Let us now see that the converse is also true.

Let \( (\mathbb{X}, \mathbb{B}, \mu) \) be a probability space. A sequence \( \{ X_k \} \) in \( \mathcal{H} = L^2(\mathbb{X}, \mathbb{B}, \mu) \) is weakly stationary if 
\[
\langle X_{t+h}, X_{t+k} \rangle = \langle X_h, X_k \rangle 
\] 
\( \forall t, h, k \in \mathbb{Z} \).

Recall also: Let \( M \subset \mathcal{H} \) nonempty. A map \( U: M \to \mathcal{H} \) such that 
\[
\langle Uf', Uf'' \rangle = \langle f', f'' \rangle \quad \forall f', f'' \in M
\]
is called isometric.

**Lemma 5** Let \( \phi: M \to \mathcal{H} \) be an isometric map. Then \( \phi \) can be extended to an isometric map on \( \mathcal{H}_M = \text{span} \{ f \in M \} \).

**Proof** Let \( f_1, \ldots, f_m \in M \) and define \( U \) for any \( f = \sum_{k=1}^m c_k f_k \) by setting
$U_f = \sum_{k=1}^{m} c_k U f_k$. Let's see that $U$ is well defined. Assume that $f = \sum_{k=1}^{m} c_k f_k = \sum_{k=1}^{m} d_k f_k$. Set $b_k = c_k - d_k$. Then

$$\left\| \sum_{k=1}^{m} b_k U f_k \right\|^2 = \left\langle \sum_{k=1}^{m} b_k U f_k, \sum_{k=1}^{m} b_k U f_k \right\rangle = \sum_{1 \leq k, \ell \leq m} b_k \overline{b_\ell} \left\langle U f_k, U f_\ell \right\rangle = \sum_{1 \leq k, \ell \leq m} b_k \overline{b_\ell} \left\langle f_k, f_\ell \right\rangle = \sum_{1 \leq k, \ell \leq m} b_k \overline{b_\ell} \left\langle U f_k, U f_\ell \right\rangle = \left\| \sum_{k=1}^{m} b_k f_k \right\|^2 = 0.$$

Let's see that the extension of $U$ defined above is isometric: let $f = \sum_{k=1}^{m} c_k f_k$ and $f' = \sum_{k=1}^{m} d_k f_k$, then

$$\left\langle U f, U f' \right\rangle = \left\langle \sum_{k=1}^{m} c_k U f_k, \sum_{k=1}^{m} d_k U f_k \right\rangle = \sum_{1 \leq k, \ell \leq m} c_k \overline{d_\ell} \left\langle U f_k, U f_\ell \right\rangle = \sum_{1 \leq k, \ell \leq m} c_k \overline{d_\ell} \left\langle f_k, f_\ell \right\rangle = \sum_{1 \leq k, \ell \leq m} c_k \overline{d_\ell} \left\langle U f_k, U f_\ell \right\rangle =$$
\[
\sum_{1 \leq k, l \leq m} c_{k,l} \langle f_k, f'_l \rangle = \langle f, f' \rangle.
\]

Any element \( f \) of \( H_m \) is a limit of elements \( f_n \) of the form \( f_n = \sum_{k=1}^{m} c_{k,n} f_k \), \( f_k \in M \).

We have: \( \lim_{n \to \infty} \| f_n - f \| = 0 \), \( \lim_{m \to \infty} \| f_n - f_m \| = 0 \), \( \lim_{n \to \infty} \| U f_n - U f_m \| = 0 \).

Recall: The linear operator \( U : H \to K \) between the Hilbert spaces \( H \) and \( K \) is unitary if it is bijective and \( \langle U f, U g \rangle_K = \langle f, g \rangle_H \) for all \( f, g \in H \).

Remark: If \( U : H \to K \) is linear and surjective, then it is unitary if \( \| U f \|_K = \| f \|_H \) for all \( f \in H \).
It follows from the polarization formula:
\[
\langle f, g \rangle = \frac{1}{4} \sum_{n=0}^{3} \| f + i^{n} g \|^{2}
\]

We can prove that any weakly stationary sequence of vectors in a Hilbert space is generated by a group of unitary operators.

**Theorem 6** Let \( \{ X_{k} \}_{k \in \mathbb{N}} \subseteq \mathcal{H} \) be weakly stationary. Let \( \mathcal{H}_{X} = \overline{\text{span}} \{ X_{k} \}_{k \in \mathbb{N}} \). There exists a unitary operator \( U : \mathcal{H}_{X} \to \mathcal{H} \) such that \( U^{n} X_{k} = X_{n+k} \).

**Proof** The formula \( U^{n} X_{k} = X_{k+n} \) defines for any \( n \in \mathbb{N} \) an isometric operator on \( X = \{ X_{k} \}_{k \in \mathbb{N}} \).

Indeed:
\[
\langle U^{n} X_{k}, U^{n} X_{k} \rangle = \langle X_{k+n}, X_{k+n} \rangle = \langle X_{k}, X_{k} \rangle
\]

By the previous lemma, the operator \( U^{n} \) can be extended to an isometric operator on \( \mathcal{H}_{X} \).
In order to show that $U^n$ is unitary, it suffices to show that $U^n$ is surjective:

$\forall f \in \mathcal{H}_x \exists f' \in \mathcal{H}_x$ such that $f = U^n f'$.

But any element $f$ of $\mathcal{H}_x$ is a limit of elements $f_s$ of the form $f_s = \sum_{l=1}^{m_s} c_{s,l} e_{k, s, l}$.

Take $f' = \lim_{n \to \infty} f_s$, where

$$f_s' = \sum_{l=1}^{m_s} c_{s,l} X_{k, s, l}$$

Remark: The family of operators $U^n$, $n \in \mathbb{Z}$ satisfies $U^n U^m = U^{n+m}$.

Let $\{X_k\}_{k \in \mathbb{Z}}$ be a weakly stationary sequence in the Hilbert space $\mathcal{H}$. By Theorem 6 and von Neumann's Mean Ergodic Theorem, the limit
\[ \Lambda(X) := \lim_{n \to \infty} \frac{X_0 + X_1 + \ldots + X_{n-1}}{n} \text{ exists in } \mathcal{H}. \]

\((X_n = U^n X_0)\)

Let's see that \(\langle \Lambda(X), X_h \rangle\) is independent of \(h\).

In fact,

\[
\langle \Lambda(X), X_h \rangle = \lim_{n \to \infty} \left\langle \frac{X_{h+1} + \ldots + X_{h+n}}{n}, X_h \right\rangle
\]

\[
= \lim_{n \to \infty} \left\langle \frac{X_{h+1} + \ldots + X_{h+n}}{n}, X_h \right\rangle = \langle \Lambda(X), X_h \rangle.
\]

This implies that

\[\langle \Lambda(X), X_h \rangle = \langle \Lambda(X), \frac{X_1 + \ldots + X_n}{n} \rangle \to \langle \Lambda(X), \Lambda(X) \rangle \text{ as } n \to \infty\]

\[\Rightarrow \langle \Lambda(X), \Lambda(X) \rangle = \| \Lambda(X) \|^2\]

i.e. \(\langle \Lambda(X), X_h \rangle = \| \Lambda(X) \|^2 \text{ for all } h \in \mathbb{Z}\)

If \(\{X_h\}_{h \in \mathbb{Z}}\) is weakly stationary, then for all \(a \in \mathbb{R}\)

\(\{e^{i \theta} X_h\}_{h \in \mathbb{Z}}\) is also weakly stationary.
\[ \Lambda(X, \theta) = \lim_{n \to \infty} \frac{e^{-i\theta_1}X_1 + e^{-i\theta_2}X_2 + \ldots + e^{-i\theta_n}X_n}{n} \]

exists in \( \mathcal{H} \). Moreover

\[ \langle \Lambda(X, \theta), e^{-i\theta}X_n \rangle \text{ is independent of } h \]

and

\[ \langle \Lambda(X, \theta), e^{-i\theta}X_n \rangle = \| \Lambda(X, \theta) \|^2 \]

We get, for \( \theta_1, \theta_2 \in \mathbb{R} \)

\[ \langle \frac{e^{-i\theta_1}X_1 + e^{-i\theta_2}X_2 + \ldots + e^{-i\theta_n}X_n}{n}, \Lambda(X, \theta_2) \rangle = \]

\[ = \frac{e^{i(\theta_2 - \theta_1)} + e^{i(\theta_2 - \theta_2)} + \ldots + e^{i(\theta_2 - \theta_n)}}{n} \cdot \| \Lambda(X, \theta_2) \|^2 \]

Since this is true \( \forall n \), \( \theta \to \infty \) we get

\[ \langle \Lambda(X, \theta_1), \Lambda(X, \theta_2) \rangle = 0 \quad \forall \theta_1, \theta_2 \in \mathbb{R} \]

i.e., the limit vectors \( \Lambda(X, \theta_1), \Lambda(X, \theta_2) \) for the twisted stationary sequences \( \{ e^{-i\theta_1}X_n \} \) and \( \{ e^{-i\theta_2}X_n \} \) are orthogonal in \( \mathcal{H} \).
An Identity for stationary sequences.

Let $S_n = X_1 + \ldots + X_n$.

We have: \textbf{Proposition 7}

\[
\frac{\|S_n\|^2}{n} + \frac{\|S_m\|^2}{m} - \frac{\|S_{n+m}\|^2}{n+m} = \]

\[
= \frac{n(n+m)}{m} \left\| \frac{S_n}{n} - \frac{S_{n+m}}{n+m} \right\|^2
\]

(Ky Fan, 1946).

\textbf{Proof.} Define $T_n, m = S_{n+m} - S_n$ so that $S_{n+m} = S_n + T_{n, m}$.

We get

\[
\left\| \frac{S_n}{n} - \frac{S_{n+m}}{n+m} \right\|^2 = \frac{\|S_n\|^2}{n^2} + \frac{\|S_{n+m}\|^2}{(n+m)^2} - \left\langle \frac{S_n}{n}, \frac{S_{n+m}}{n+m} \right\rangle
\]

\[
- \left\langle \frac{S_{n+m}}{n+m}, \frac{S_n}{n} \right\rangle \quad \text{and}
\]

\[
\frac{n(n+m)}{m} \left\| \frac{S_n}{n} - \frac{S_{n+m}}{n+m} \right\|^2 = \frac{(n+m)\|S_n\|^2}{nm} + \frac{n\|S_{n+m}\|^2}{m(n+m)}
\]

\[
- \frac{1}{m} \left( \left\langle S_n, S_{n+m} \right\rangle + \left\langle S_{n+m}, S_n \right\rangle \right)
\]

= \]
\[
\begin{align*}
\frac{\|S_n\|^2}{n} + \left(\frac{\|S_n\|^2}{m} - \frac{\|S_m\|^2}{m}\right) + \frac{\|S_m\|^2}{m} - \frac{\|S_{n+m}\|^2}{n+m} \\
+ \frac{\|S_{n+m}\|^2}{m} - \frac{1}{m} \left(\langle S_n, S_{n+m} \rangle + \langle S_{n+m}, S_n \rangle\right).
\end{align*}
\]

However:
\[
\langle S_n, S_{n+m} \rangle + \langle S_{n+m}, S_n \rangle = 2\|S_n\|^2 + \langle S_n, T_n, m \rangle + \langle T_n, m, S_n \rangle
\]

and this yields
\[
\frac{n(n+m)}{m} \left\| \frac{S_n}{n} - \frac{S_{n+m}}{n+m} \right\|^2 = \frac{\|S_n\|^2}{n} + \frac{\|S_m\|^2}{m} - \frac{\|S_{n+m}\|^2}{n+m} + \\
+ \frac{1}{m} \left(\|S_{n+m}\|^2 - \|S_n\|^2 - \|S_m\|^2 - \langle S_n, T_n, m \rangle - \langle T_n, m, S_n \rangle\right)
\]
\[
= \frac{\|S_n\|^2}{n} + \frac{\|S_m\|^2}{m} - \frac{\|S_{n+m}\|^2}{n+m} \quad \text{(can be simplified)}
\]

A simple consequence of Ky Fan's Identity is:
\[
\frac{\|S_{n+m}\|^2}{n+m} \leq \frac{\|S_n\|^2}{n} + \frac{\|S_m\|^2}{m}
\]
i.e. the sequence \( g_n = \frac{\|S_n\|_2^2}{n} \) is subadditive.

**Lemma 8** If \( \{g_n\}_{n=1}^\infty \) is a subadditive sequence of real numbers, then \( \frac{g_n}{n} \) converges to \( \inf_{n \geq 1} \frac{g_n}{n} \) as \( n \to \infty \).

**Proof** Fix \( N > 0 \). Write \( n = j_n \cdot N + r_n \) with \( 1 \leq r_n \leq N \). Clearly \( \frac{j_n \cdot N}{n} \to \frac{1}{2} \) as \( n \to \infty \).

Moreover

\[
\inf_{n \geq 1} \frac{g_n}{n} \leq \frac{g_n}{n} \leq \frac{g_{j_n \cdot N}}{j_n \cdot N} + \frac{g_{r_n}}{n} \leq \frac{j_n \cdot g_N}{j_n \cdot N} + \frac{g_{r_n}}{n} = \]

\[
= \frac{g_N}{N} + \frac{g_{r_n}}{n}.
\]

Take \( n \to \infty \) and get

\[
\inf_{n \geq 1} \frac{g_n}{n} \leq \limsup_{n \to \infty} \frac{g_n}{n} \leq \limsup_{n \to \infty} \frac{g_N}{n} \leq \frac{g_N}{N}.
\]

\( N \) was arbitrary so the same is proven. \( \square \)
This lemma and the identity by Ky Fan applied to
\[ g_n = \frac{\|S_n\|^2}{n} \]

imply that for a stationary sequence \( \{X_k\}_{k \in \mathbb{Z}} \subset \mathcal{H} \) we have that

\[
\lim_{n \to \infty} \left\| \frac{S_n}{n} \right\| = \inf_{n \geq 1} \left\| \frac{S_n}{n} \right\|
\]

where \( S_n = X_0 + \ldots + X_{n-1} \)

In particular the above formula holds when

\[ X_k = \mathbb{1} \mathbf{T}^k \]

where \( f \in L^2(X, \mathcal{B}, \mu) \)

and \( T: X \to X \) \( \mu \)-preserving

\( \left( \frac{S_n}{n} = A_n^T f \text{ in this case. Recall older notation} \right) \)

\( \text{(We know already that } A_n^T f \to \mathbb{1} f = \mathbb{1}(f | T) \text{ by von Neumann's T.D.)} \)