Examples of dynamical systems: the doubling map, coding and conjugacies.

Let \( T(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \leq x < 1 \end{cases} \)

If we view \( (0,1) \) as \( \mathbb{R}/\mathbb{Z} \) or \( S^1 \), we notice that the map \( T \) is well defined: \( T(0) = T(1) = 0 \). In multiplicative notation, \( T(e^{2\pi i \theta}) = e^{2\pi i 2\theta} = e^{2\pi i 6\theta/2} \).

We can see that \( T \) is continuous on \( S^1 \) and \( T \) is not invertible.

\[ T^{-1}(y) = \left\{ y, \frac{y}{2} + \frac{1}{2} \right\}. \]

Moreover: \( T \) is expanding: if \( d(x, y) < \frac{1}{4} \)

Then \( d(T(x), T(y)) = 2d(x, y) \).

Questions: Are there periodic points? Are there points with dense orbit?
To answer these questions we use general facts from the topological side of Ergodic theory.

Let \( T: X \to X \) and \( S: Y \to Y \) two maps.

A conjugacy between \( T \) and \( S \) is an invertible map \( \psi: Y \to X \) such that \( \psi \circ S = T \circ \psi \).

In other words, the diagram commutes:

\[
\begin{array}{ccc}
Y & \xrightarrow{S} & Y \\
\downarrow{\psi} & & \downarrow{\psi} \\
X & \xrightarrow{T} & X
\end{array}
\]

**Lemma 1.3** Let \( T \) and \( S \) be conjugated by \( \psi \). Then \( y \) is periodic of period \( n \) for \( S \) if and only if \( \psi(y) \) is periodic of period \( n \) for \( T \).

**Proof**

One can check (by induction) that if \( \psi \circ S = T \circ \psi \), then \( \psi \circ S^n = T^n \circ \psi \). If \( \psi \) is invertible, then \( S^n = \psi \circ T^n \circ \psi^{-1} \).

Let \( S^n(y) = y \), then \( T^n(\psi(y)) = \psi(S^n(y)) = \psi(y) \).

Conversely, let \( T^n(\psi(y)) = \psi(y) \). Then

\[
S^n(y) = \psi^{-1}(T^n(\psi(y))) = \psi^{-1}(\psi(y)) = y
\]
Def: a semi-conjugacy between $T$ and $S$ is a surjective map $\psi: Y \to X$ such that $\psi \circ S = T \circ \psi$.

We say that $S: Y \to Y$ is an extension of $T: X \to X$ and that $T: X \to X$ is a factor of $S: Y \to Y$.

Lemmata. Let $T$ and $S$ be semi-conjugated by $\psi$.

If $y$ is a periodic point of period $n$ for $S$, then $\psi(y)$ is a periodic point of period $n$ for $T$.

The converse statement is in general not true.

Back to the doubling map $T: [0,1) \to [0,1)$, $T(x) = 2x \mod 1$. Consider the binary expansion of $x \in [0,1)$:

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}, \quad x_i \in \{0,1\}.$$

Then $T(x) = \sum_{i=1}^{\infty} \frac{2x_i}{2^{i+1}} \mod 1 = \sum_{i=2}^{\infty} \frac{x_i}{2^{i-1}} = \sum_{i=1}^{\infty} \frac{x_{i+1}}{2^i}$.

In other words, $T$ shifts the digits by 1.

Let $\Sigma^+ = \{0,1\}^\infty$ and $a = (a_i)_{i \geq 1} \in \Sigma^+$.

The shift map is $\sigma: \Sigma^+ \to \Sigma^+$, $\sigma(a) = b$ where $b = (b_i)_{i \geq 1}$ and $b_i = a_{i+1}$.

Notice that $\sigma$ is not invertible.
Let \( \psi : \Sigma^+ \rightarrow [0,1] \) be the map 
\[
\psi((a_i)_{i \geq 1}) = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \in [0,1].
\]
Notice that \( \psi \) is well defined and onto, but not 1-1.

**Proposition 1.4**

\( \psi \) is a semi-conjugacy between \( \sigma : \Sigma^+ \rightarrow \Sigma^+ \) and the doubling map \( T : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} \).

Proof: Since \( \psi \) is onto, we need to show that

\[
\Sigma^+ \xrightarrow{T} \Sigma^+ \text{ commutes.}
\]

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
\psi \\
\psi \\
\psi
\end{array}
\begin{array}{c}
[0,1] \\
[0,1] \\
[0,1]
\end{array}
\begin{array}{c}
= \sum_{i=1}^{\infty} \frac{a_i}{2^i} \\
= \sum_{i=1}^{\infty} \frac{b_i}{2^i}
\end{array}
\]

In fact, \( \psi(\sigma((a_i)_{i \geq 1})) = \psi((b_i)_{i \geq 1}) = \sum_{i=1}^{\infty} \frac{b_i}{2^i} = \sum_{i=1}^{\infty} \frac{a_i+1}{2^i} \)

and
\[
T(\psi((a_i)_{i \geq 1})) = T\left(\sum_{i=1}^{\infty} \frac{a_i}{2^i}\right) = \sum_{i=1}^{\infty} \frac{a_i}{2^{i+1}} \text{ mod 1} = \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^{i+1}}.
\]

Now we use periodic points for \( \sigma : \Sigma^+ \rightarrow \Sigma^+ \) to produce periodic points for \( T \). (Via Lemma 1.3')
Theorem 1.5  

(a) \( T : [0,1] \to [0,1] \)  
\[ \text{Tax} = 2x \mod 1 \]

has \( 2^n - 1 \) periodic points of period \( n \).

(b) The set of periodic points is dense in \([0,1]\).

Proof:  

(a) By Proposition 1.4, periodic points of period \( n \) for \( \phi : \Sigma^+ \to \Sigma^+ \) are mapped to periodic points of period \( n \) for \( T \).

What are periodic points of period \( n \) for \( \Sigma^+ \)?

prescribed by specifying \( n \) binary digits \( \Rightarrow \)

\( 2^n \) choices. However, \( \phi([0, \ldots, 0, \ldots]) = 0 = 1 = \phi([1, 1, \ldots, 1, \ldots]) \mod 1 \)

and these are the only two periodic points that give the same image under \( \phi \).

(b) Exercise

Symbolic coding for the doubling map.

Consider the partition \( A_0 \sqcup A_1 \) of \([0,1]\):

\( A_0 = [0, \frac{1}{2}) \), \( A_1 = (\frac{1}{2}, 1) \)

Define \( \phi : [0,1] \to \Sigma^+ \) \( \phi(x) = (a_i)_{i \geq 0} \)

where \( a_i = \begin{cases} 0 & \text{if } T^i(x) \in A_0 \\ 1 & \text{if } T^i(x) \in A_1 \end{cases} \)
the sequence \((a_i)_{i=0}^\infty\) is the itinerary of \(\Omega_T^+(x)\) w.r.t. the partition \(\{A_0, A_1, \ldots\}\):

\[x \in A_{a_0}, \ T(x) \in A_{a_1}, \ T^2(x) \in A_{a_2}, \ldots.\]

**Proposition 1.6** If \((a_i)_{i=0}^\infty\) is the itinerary of \(x \in [0, 1]\)
then \(x = \sum_{i=1}^{\infty} \frac{a_i - 1}{2^i}\), i.e. \((a_i)_{i=0}^\infty\) is one of the two binary expansions of \(x\).

**Proof**

If \(a_0 = 0\), then \(0 \leq x < \frac{1}{2}\)
If \(a_0 = 1\) then \(\frac{1}{2} \leq x \leq 1\)

Say we want to look at \(k\)th entry \(a_k\).

If \(x_1, x_2, \ldots, x_k, \ldots\) are \(n\) digits of binary expansion
of \(x\), then the digits of \(T^k(x)\) are \(x_{k+1}, x_{k+2}, \ldots\)
and the itinerary of \(T^k(x)\) is \(a_k, a_{k+1}, a_{k+2}, \ldots\)
then reason as above

\(a_k = 0 \Rightarrow T^k(x) \in A_0\)
\(a_k = 1 \Rightarrow T^k(x) \in A_1\)

**Corollary** \(\psi\) is a right inverse for \(\psi \circ \phi: \mathbb{R}_2 \to \mathbb{R}_2\) in the identity map
Now fix \( a_0, a_1, \ldots, a_n \in \mathbb{Z}_0 \mathbb{R} \). Set

\[
I(a_0, a_1, \ldots, a_n) = \{ x \in (0, 1) \mid \phi(x) = (a_0, a_1, \ldots, a_n, \ldots) \} = \{ x \in (0, 1) \mid T^k(x) \in P_{a_k} \text{ for } 0 \leq k \leq n \}\]

Notice that

\[ I(a_0, a_1, \ldots, a_n) = P_{a_n} \cap T^{-1}(P_{a_{n-1}}) \cap \cdots \cap T^{-n}(P_{a_1}) = P_{a_0} \cap (0, 1). \]

Recall \( A_0 = [0, \frac{1}{2}) \), \( A_1 = (\frac{1}{2}, 1) \).

\[
T^{-1}(A_0) = [0, \frac{1}{4}) \cup [\frac{1}{2}, \frac{3}{4})
\]

\[
T^{-1}(A_1) = (\frac{1}{4}, \frac{1}{2}) \cup [\frac{3}{4}, 1)
\]

\[
I(0, 0) = [0, \frac{1}{4}), I(0, 1) = (\frac{1}{4}, \frac{1}{2}), I(1, 0) = (\frac{1}{2}, \frac{3}{4}) \), I(1, 1) = (\frac{3}{4}, 1).
\]

In general \( I(a_0, \ldots, a_n) \) is an interval of length \( 2^{-(n+1)} \).

We have a partition

\[
\bigcup \{ I(a_0, a_1, \ldots, a_n) \mid (a_0, a_1, \ldots, a_n) \in \{0, 1\}^n \}
\]

and each \( I(a_0, \ldots, a_n) \) is of the form \( [\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}) \) for \( 0 \leq k < 2^{n+1} \).

Let's construct a dense subset for \( T \).
Theorem 1.7  Let $T(x) = 2x$ not 1 on $[0,1)$. There exists a point $\tilde{x}$ s.t. $\mathcal{O}^+_{\tilde{x}}(\tilde{x})$ is dense in $[0,1)$.

**Proof**  Enough to show that $\forall n \geq 1$, $\mathcal{O}^+_{\tilde{x}}(\tilde{x})$ visits all intervals of the form $I(a_0, a_1, \ldots, a_n)$.

In fact $\forall y \in [0,1)$, $\forall \varepsilon > 0$, take $N$ large enough s.t. $2^{-N+1} \leq \varepsilon$ and take $I(a_0, \ldots, a_N) \ni y$.

If we showed that $\exists k \geq 0$ s.t. $T^k(\tilde{x}) \in I(a_0, \ldots, a_N)$, then $d(T^k(\tilde{x}), y) < 2^{-N+1} \leq \varepsilon \Rightarrow \mathcal{O}^+_{\tilde{x}}(\tilde{x})$ is dense.

Define

$$\left(\tilde{a}_i\right)_{i \geq 0} = \left(0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, \ldots\right)$$

- all possible words of length 1
- all possible words of length 2
- all possible words of length 3
- \ldots

To see that the orbit of $\tilde{x} = \psi\left(\left(\tilde{a}_i\right)_{i \geq 0}\right)$ visits $I(a_0, \ldots, a_n)$ first where the word $a_0, \ldots, a_n$ appears in $\left(\tilde{a}_i\right)_{i \geq 0}$, i.e.

$$\tilde{a}_k = a_0, \quad \tilde{a}_{k+1} = a_1, \quad \ldots, \quad \tilde{a}_{k+n} = a_n$$

and $T^k(\tilde{x}) \in I(\tilde{a}_k, \ldots, \tilde{a}_{k+n}) = I(a_0, \ldots, a_n)$.
Remark: The map \( x \mapsto 3x \mod 1 \) is conjugated to the shift over \( \{0, 1, 2\} \mathbb{N} \) and the same construction holds in this case.

Same for \( x \mapsto mx \mod 1 \), \( m \in \mathbb{N} \) and the shift over \( \{0, \ldots, m-1\} \mathbb{N} \).

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Barker's map as extension of doubly map.

\[ X: [0, 1)^2 \to [0, 1)^2 \quad F: X \to X \]

\[ F(x, y) = \begin{cases} (2x, y/2), & 0 \leq x < \frac{1}{2} \\ (2x-1, \frac{y+1}{2}), & \frac{1}{2} \leq x < 1 \end{cases} \]

Notice the stretching in the horizontal direction by factor 2 and the contraction in the vertical one by factor 2.

\( F \) is invertible:

\[ F^{-1}(x, y) = \begin{cases} (\frac{x}{2}, 2y), & 0 \leq y < \frac{1}{2} \\ (\frac{x+1}{2}, 2y-1), & \frac{1}{2} \leq y < 1 \end{cases} \]
Proposition 1.8  Fix an invertible extension of the doubling map $T$

Proof. We already know $F$ is invertible. Let $\pi : [0,1) \to \{0,1\}$ be the projection onto the 1st coordinate.

Claim: $F$ and $T$ are semi-conjugated by $\pi$:

$\pi : [0,1)^2 \to [0,1]^2$

$\pi \downarrow \quad \quad \downarrow \pi$

$[0,1) \to [0,1)$

In fact: $\pi(F(x,y)) = 2x \mod 1 = T(x) = T(\pi(x,y))$.

We already know that $T$ is semi-conjugated to the shift $\sigma^+: \Sigma^+ \to \Sigma^+$, $\Sigma^+ = \{0,1\}^\mathbb{N}$.

Let's see that $F$ is semi-conjugated to the two-sided shift $\sigma : \Sigma \to \Sigma$, $\Sigma = \{0,1\}^\mathbb{Z}$.

$a = (a_i)_{i \in \mathbb{Z}} \in \Sigma$

$\sigma(a) = b \quad , \quad b = (b_i)_{i \in \mathbb{Z}} \quad , \quad b_i = a_{i+1}$

[Notice that $\sigma$ is invertible]

Let's use the idea of coding as before.

Let $A_0 = [0,\frac{1}{2}) \times [0,1)$, $A_1 = (\frac{1}{2},1) \times [0,1)$.
The bi-infinite itinerary of \((x,y)\) w.r.t. the partition 
\([A_0, A_1]\) is the sequence 
\((a_i)_{i \in \mathbb{Z}} \in \Sigma\) defined as
\[
    a_k = \begin{cases} 
    0 & \text{if } F^k(x,y) \in A_0 \\
    1 & \text{if } F^k(x,y) \in A_1 
    \end{cases}
\]

In particular if \(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\)
is the itinerary of \(\mathcal{O}_F((x,y))\), then
\[
F^k(x,y) \in A_{a_k} \quad \forall \ k \in \mathbb{Z}.
\]

Given \(m, n \in \mathbb{Z}\) and \(a_n \in \{0, 1\}\) for \(-m \leq k \leq n\),
\[
A_{-m,n}(a_{-m}, \ldots, a_n) = \big\{ (x,y) \in X : F^k(x,y) \in A_{a_k} \big\}
\]
for \(-m \leq k \leq n\).

Set of points in \(X\) whose itinerary between \(-m\) and \(n\)
agrees with \(a_{-m}, \ldots, a_n\).

Notice that
\[
A_{-m,n}(a_{-m}, \ldots, a_n) = \bigcap_{k=-m}^{n} F^{-k}(R_{a_k})
\]
Ex: $F^{-1}(A_0) = \quad$ 

$A_{0,1}(1,0) = A_1 \cap F^{-1}(A_0) = \quad$ 

In pencil

$R_{0,n}(a_0, a_1, \ldots, a_n) = I(a_0, \ldots, a_n) \times [0,1) = \quad$ (future)

$= \left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right) \times [0,1) = \quad$

for some $0 \leq k < 2^{n+1}$.

What about the points sharing the same itinerary in the past?

$F(A_0) = \quad$ 

$F^2(A_1) = F(\quad) = \quad$

and $A_{-2,-1}(0,1) = F^2(A_0) \cap F(A_1) = \quad$

In pencil

$A_{-n,-1}(a_n, \ldots, a_1)$ is a horizontal rectangle of the form
Theorem 1.9

The Baker map $F$ is semi-conjugated to the full shift $\sigma: \Sigma \to \Sigma$ via

$\Psi: \Sigma \to [0,1)^2$,

$\Psi((a_i)_{i \in \mathbb{Z}}) = (x,y)$, where

$x = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$, \quad $y = \sum_{i=1}^{\infty} \frac{a_{-i}}{2^i}$

["Future" $(a_i)_{i \geq 0}$ determines the binary expansion of $x$ while the "past" $(a_i)_{i \leq 0}$ determines the binary expansion of $y"$]

Proof: $\Psi$ is well defined and onto. Let us check that $\Psi_\sigma = F \Psi$:

$\Psi(\sigma((a_i)_{i \in \mathbb{Z}})) = \Psi((a_{i+1})_{i \in \mathbb{Z}}) = \left(\sum_{i=1}^{\infty} \frac{a_i}{2^i}, \sum_{i=1}^{\infty} \frac{a_{-i+1}}{2^i}\right)$
Note that if \( a_0(x) \) is the first entry of the itinerary of \( x \), then \( a_0(x) = \begin{cases} 0 & \text{if } (x,y) \in A_0 \\ 1 & \text{if } (x,y) \in A_1 \end{cases} \)

and therefore \( T(x) \)

\[
F(x,y) = \left( 2x \mod 1, \frac{y + a_0(x)}{2} \right).
\]

This implies:

\[
F(\Phi((a_i)_{i \in \mathbb{Z}})) = F\left( \sum_{i=1}^{\infty} \frac{a_{i-1}}{2^i}, \sum_{i=1}^{\infty} \frac{a_i}{2^i} \right) = (T(\sum_{i=1}^{\infty} \frac{a_{i-1}}{2^i}), \frac{a_0}{2} + \frac{1}{2} \sum_{i=1}^{\infty} \frac{a_i}{2^i}) = \left( \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i}, \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i} \right)
\]

That is \( \Phi \circ \delta = \delta \circ \Phi \).

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Bigger picture

\[
\sum \xrightarrow{\delta} \sum \xrightarrow{\Phi} \sum \xrightarrow{\pi} \mathbb{C} \xrightarrow{T} \mathbb{C} \xrightarrow{\pi} \sum \xrightarrow{\Psi} \sum^+ \xrightarrow{\Theta^+} \sum^+ \]
\[
\Phi \xrightarrow{(0,1)^2} F \xrightarrow{(0,1)^2} \Phi
\]

\[
\pi \xrightarrow{T} \pi
\]

\[
\Psi \xrightarrow{T} \Psi
\]