It is known that every \( x \in (0,1) \) can be written as
\[
x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \quad \text{with } a_i \in \mathbb{N}
\]

Clearly, if \( \ast \) is a finite expression, then \( x \) is rational. Let's see the converse.

Let \( a, b \in \mathbb{Z} \), \( a > b > 0 \)

Euclidean algorithm to compute \( \gcd(a, b) \)

\[
r_0 = a, \quad r_1 = b
\]

Determine \( a_1, r_1 \), and \( r_2 > 0 \) so

\[
r_0 = a_1 r_1 + r_2, \quad 0 \leq r_2 < r_1
\]

If \( r_2 \neq 0 \) repeat this procedure. The algorithm clearly stops after at most \( r_1 \) steps.

\[
r_k = a_k r_{k+1} + r_{k+2} \quad \text{for } k \leq n-1
\]

\[
0 = r_{n+1} < r_n < \cdots < r_2 < r_1 \quad \text{and } r_n = \gcd(a, b)
\]
Notice that
\[ a_1 = \left[ \frac{r_0}{r_1} \right], \quad a_2 = \left[ \frac{r_2}{r_1} \right], \quad \ldots, \quad a_n = \left[ \frac{r_{n-1}}{r_n} \right] \]

Set
\[ x = x_0 = \frac{b}{a} = \frac{r_1}{r_0}, \quad x_1 = \frac{r_2}{r_1}, \quad x_2 = \frac{r_3}{r_2}, \quad \ldots, \quad x_{n-1} = \frac{r_n}{r_{n-1}} \]

we get
\[
- \frac{1}{x} = a_1 + x_1 \\
- \frac{1}{x_1} = a_2 + x_2 \\
- \frac{1}{x_2} = a_3 + x_3 \\
\vdots \\
- \frac{1}{x_{n-2}} = a_{n-1} + x_{n-1} \\
- \frac{1}{x_{n-1}} = a_n + 0
\]

\[ x = \frac{1}{a_1 + x_1} = \frac{1}{a_1 + \frac{1}{a_2 + x_2}} = \frac{1}{\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{ \ldots }}}} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{ \ldots } + \frac{1}{a_n}}}} \]

i.e. every rational has a finite continued fraction expansion, denoted
\[ [a_1, a_2, \ldots, a_n] \]
Notice that it is not unique:
\[
[a_1, a_2, \ldots, a_n] = [a_1, \ldots, a_{n-1}, 1]
\]
Define the map \( T: [0,1) \to [0,1) \)
\[
T(x) = \begin{cases} 
0, & x = 0 \\
\frac{1}{x - \lfloor \frac{1}{x} \rfloor} = \left\lfloor \frac{1}{x} \right\rfloor, & x \neq 0
\end{cases}
\]
This is called the Gauss map.

Let \( x \in (0,1) \backslash \mathbb{Q} \) now. Set \( x_0 = x \)
\( x_1 = T(x_0), \ x_2 = T(x_1), \ldots \)
We put \( x_n \in (0,1) \backslash \mathbb{Q} \) for all \( n > 0 \).
Let \( a_n = a_n(x) = \left\lfloor \frac{1}{x_{n-1}} \right\rfloor \) \( n \geq 1 \).

We put:
\[
x = \frac{1}{a_1 + x_1} = \frac{1}{a_1 + \frac{1}{a_2 + x_2}} = \ldots = \frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{x_n}}} = [a_1, a_2, \ldots, a_{n-1}, a_n+x_n]
\]
and by \( x = [a_1, a_2, \ldots] \) we mean

\[
x = \lim_{n \to \infty} [a_1, a_2, \ldots, a_n].
\]

(We will see that this limit exists.)

By truncating \( x = [a_1, a_2, \ldots] \) we get the **convergents** of \( x \)

\[
\frac{p_n}{q_n} = [a_1, \ldots, a_n] \in \mathbb{Q}.
\]

**Example**

\[
\phi = \frac{-1 + \sqrt{5}}{2}, \quad \phi + \phi + 1 = 0 \quad \text{hence}
\]

\[
\phi = \frac{1}{1 + \phi} = \frac{1}{1 + \frac{1}{1 + \phi}} = \ldots = [1, 1, 1, \ldots] = [\frac{1}{2}]
\]

\[
x = \sqrt{2} - 1, \quad x = \frac{1}{2 + x} = \frac{1}{2 + \frac{1}{2 + x}} = \ldots = [2, 2, \ldots] = [\frac{1}{2}]
\]

**Exercise**: Every \( x \) with an eventually periodic continued fraction expansion is the root of a polynomial of degree 2.
Remark. The converse is also true (Lagrange's Theorem, 1770).

Properties of convergents

Let \( \frac{p_n}{q_n} = [a_1, \ldots, a_n] \).

Lemma

\[
\begin{pmatrix} p_n & P_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}
\]

where \( p_{-1} = q_0 = 1 \) and \( p_0 = q_{-1} = 0 \).

Proof. By induction (exercise).

Corollary

\( p_{n+1} = a_{n+1} p_n + p_{n-1} \), \( q_{n+1} = a_{n+1} q_n + q_{n-1} \).

Proof

\[
\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & P_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}
\]

from the lemma.

We get a strictly increasing sequence of denominators

\( 1 = q_0 < q_1 < q_2 < \ldots < q_n < \ldots \)

and, since \( q_n = a_n q_{n-1} + q_{n-2} \geq 2q_{n-2} \), we get
that $q_n > 2^{\frac{n-2}{2}}$ for every $x = [a_1, a_2, \ldots]$. 

Another consequence of the lemma is the corollary

$$\sum p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}.$$ 

This yields:

$$\frac{p_1}{q_1} = \frac{1}{q_0 q_1}, \quad \frac{p_2}{q_2} = \frac{p_1}{q_1} - \frac{1}{q_1 q_2} = \frac{1}{q_0 q_1 q_2},$$

$$\frac{p_3}{q_3} = \frac{p_2}{q_2} + \frac{1}{q_2 q_3} = \frac{1}{q_0 q_1 q_2} + \frac{1}{q_2 q_3},$$

$$\frac{p_n}{q_n} = \frac{p_{n-1}}{q_{n-1}} + (-1)^{n+1} \frac{1}{q_{n-1} q_n} = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{q_i q_{i-1}}.$$ 

This means that $[a_1, a_2, \ldots]$ is not just a formal expression:

$$[a_1, a_2, \ldots] = \lim_{n \to \infty} [a_1, \ldots, a_n] = \lim_{n \to \infty} \frac{p_n}{q_n} = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{q_{i-1} q_i}$$

and this series converges absolutely since $q_i > 2^{\frac{i-2}{2}}$. 

Le
Another consequence of the lemma is the following ordering of the convergents:

\[
\frac{P_0}{Q_0} < \frac{P_2}{Q_2} < \frac{P_4}{Q_4} < \ldots < x < \ldots < \frac{P_5}{Q_5} < \frac{P_3}{Q_3} < \frac{P_1}{Q_1},
\]

where all the inequalities are strict for \(x \notin \mathbb{Q}\).

Let us now see how the Gauss map can be interpreted geometrically in terms of circle rotations:

Let \(S' = [0, 1) = \bigcirc\) and consider the rotation \(R_\theta\) by angle \(\theta\).

Find \(k_1\) such that

\[k_1 \theta \leq 1 < (k_1 + 1) \theta,\]

i.e., \(S'\) can be covered by \((k_1 + 1)\) arcs of length \(\theta\) but not by \(k_1\) arcs.

From our choice of \(k_1\), we see that

\[k_1 \leq \frac{1}{\theta} < (k_1 + 1),\]

i.e., \(k_1 = \left\lfloor \frac{1}{\theta} \right\rfloor\), i.e., \(k_1\) is the first entry of the C.F. expansion of \(\theta\).
\( \alpha = [k_1, k_2, k_3, \ldots] \) for some \( k_2, k_3, \ldots \in \mathbb{N} \).

Let \( \Delta^{(0)}_1 = [0, \alpha] \), \( \Delta^{(0)}_i = R_{\alpha}^{i-1}(\Delta^{(0)}_1), 1 \leq i \leq k_1 \). These are non-overlapping intervals.

Let \( \Delta^{(1)}_1 \) be the remaining interval after covering \([0, 1]\) with \( \Delta^{(0)}_i, 1 \leq i \leq k_1 \):

\[
\Delta^{(1)}_1 = [k_1 \alpha, 1] = [1 - \alpha k_1, 0]
\]

Let us consider the ratios

\[
\left| \frac{\Delta^{(1)}_1}{\Delta^{(0)}_1} \right| = \frac{1 - k_1 \alpha}{\alpha} = \frac{1 - \frac{k_1}{k_1 + [k_2, k_3, \ldots]}}{1} = \frac{k_2, k_3, \ldots}{k_1 + [k_2, k_3, \ldots]} = T(\alpha)
\]
Where \( T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \{\frac{1}{x}\} \) is the Gauss map.

Some iterates of \( \Delta^{(i)} \) under \( R_a \) will look like these:

Let's consider these intervals of length \(|\Delta^{(i)}|\), adjacent to the right endpoint of \( \Delta^{(o)} \), and to the right of 0. Let \( m \) be their number:

\[ m \cdot |\Delta^{(i)}| \leq |\Delta^{(o)}| < (m+1) |\Delta^{(i)}|, \quad \text{i.e.} \]

\[ m \leq \frac{|\Delta^{(o)}|}{|\Delta^{(i)}|} = \frac{1}{T(a)} < m+1, \quad \text{i.e.} \]

\[ m = \left\lfloor \frac{1}{T(a)} \right\rfloor = k_2 \quad \text{(the second entry in the C.F. expansion of } a) \]

Let \( \Delta^{(2)} \) be the remaining one to the right of 0.
We have
\[
\frac{|\Delta_1^{(2)}|}{|\Delta_1^{(1)}|} = \frac{|\Delta_1^{(2)}|}{|\Delta_1^{(0)}|} = \frac{d - k_2 \frac{|\Delta_1^{(1)}|}{|\Delta_1^{(0)}|}}{T(d)}
\]
\[
= 1 - \frac{k_2}{k_2 + [k_3, k_4, \ldots]}\frac{1}{k_2 + [k_3, k_4, \ldots]} = [k_3, k_4, \ldots] = T_2(d).
\]

and so on. We see that the sequence of partial quotients \(k_1, k_2, k_3, \ldots\) is constructed by means of a geometric Euclidean algorithm.
this shows how \( k_1, k_2, \ldots \) are constructed by a geometric version of the Euclidean algorithm.

Assume we have constructed \( \Delta_i^{(n-1)} \) and \( \Delta_i^{(n)} \) with 0 as a common endpoint and such that

\[
\frac{|\Delta_i^{(n)}|}{|\Delta_i^{(n-1)}|} = T_n^{(\alpha)} = [k_{n+1}, k_{n+2}, \ldots].
\]

Starting at the endpoint of \( \Delta_i^{(n-1)} \) (other than 0), construct as many intervals of length \( |\Delta_i^{(n)}| \) as possible, and denote the residual interval by \( \Delta_i^{(n+1)} \). If \( m = \# \) of such intervals,

then \( m \cdot |\Delta_i^{(n)}| \leq |\Delta_i^{(n-1)}| < (m+1) \cdot |\Delta_i^{(n)}| \),

or

\[
m = \left| \frac{|\Delta_i^{(n-1)}|}{|\Delta_i^{(n)}|} \right| = \left| \frac{1}{T_n^{(\alpha)}} \right| = k_{n+1},
\]

and

\[
\frac{|\Delta_i^{(n+1)}|}{|\Delta_i^{(n)}|} = \frac{|\Delta_i^{(n-1)}| - k_n |\Delta_i^{(n)}|}{|\Delta_i^{(n)}|} = \frac{1}{T_n^{(\alpha)}} - \frac{1}{|T_n^{(\alpha)}|} = \frac{1}{T_n^{(\alpha)}} \left( \sum_{n=0}^{\infty} \right) = T_n^{(\alpha)}.
\]
Claim: The arcs $\Delta_1^{(n)}$ have endpoints $0$ and $R_\alpha q_n(0)$.

Proof. By induction, assume it is true for $\Delta_1^{(n-1)}$ and $\Delta_1^{(n)}$. By construction:

- $\Delta_1^{(n-1)}$
- $\Delta_1^{(n)}$
- $0$
- $R_\alpha q_n(0)$
- $R_\alpha q_n+1q_n(\Delta_1^{(n)})$
- $\cdots$
- arcs of length $|\Delta_1^{(n)}|$, given by:
- $R_\alpha q_n-1(D_1^{(n)})$
- $R_\alpha q_n+1q_n(D_1^{(n)})$
- $R_\alpha q_n+2q_n(D_1^{(n)})$
- $\cdots$
- $R_\alpha q_n+(k_n-1)q_n(D_1^{(n)})$

and the arc $R_\alpha q_n+1k_nq_n(D_1^{(n)}) = R_\alpha q_n(D_1^{(n)})$ covers $0$. Its endpoints are $0$ and $R_\alpha q_n+1(0)$.

Let $\Delta_1^{(n)} = R_\alpha q_n^{-1}(\Delta_1^{(n)})$. 


Δ(n) = R⁻¹(D(0))

0 and q_2 = R_q(0)

Δ_i = R_i⁻¹(Δ(0))

Δ_{i+1} = k_i q_{n+1} + q_{n-1}

A_i have endpoints

q_0 = 1, q_{n+1} = 1

q_i = q_{i+1} = 1
Theorem

The collection of intervals $A_i^{(n-1)}$, $1 \leq i \leq q_n$, $A_j^{(n)}$, $1 \leq j \leq q_n$, satisfy the properties:

1. $A_i^{(n-1)} \cap A_i^{(n-1)} = \emptyset$ for $i_1 \neq i_2$  
   [See Figure in the previous page]
2. $A_{i_1}^{(n)} \cap A_{i_2}^{(n)} = \emptyset$ for $j_1 \neq j_2$
3. $A_i^{(n-1)} \cap A_j^{(n)} = \emptyset$ for all $i, j$
4. $\bigcup_{i=1}^{q_n} A_i^{(n-1)} \cup \bigcup_{j=1}^{q_n} A_j^{(n)} = [0, 1)$ (modulo 0).

Proof. See ["Topics in Ergodic Theory" by Sinai]

Let us see how to define the intervals $R_n^{q_{n-1}+j} A_j^{(n)}$ for the Farey fractions:

| $0/0$ | $1/1$ |
| $0/1$ | $1/2$ |
| $1/0$ | $1/1$ |
| $1/2$ | $2/3$ |
| $1/3$ | $2/5$ |
| $1/4$ | $3/5$ |
| $1/5$ | $3/6$ |
| $1/6$ | $4/7$ |
| $1/7$ | $5/9$ |
Let the intervals $J_j^{(i)}$ be as below:

Define the following symbolic coding for $x \in (0, 1) \setminus \mathbb{Q}$.

$$f_i(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ 1 & \text{if } x \geq \frac{1}{2} \end{cases}$$

Let $f_1(x), f_2(x), \ldots, f_{n-1}(x) \in \{0, 1\}$ be already defined. Then $x \in I_{n-1}^{(n-1)}$ for some $k$. We have $I_{n}^{(n-1)} = I_{k}^{(n)} \cup I_{k+1}^{(n)}$. Set

$$f_n(x) = \begin{cases} 0 & \text{if } x \in I_{k}^{(n)} \\ 1 & \text{if } x \in I_{k+1}^{(n)} \end{cases}$$
Binary tree structure

The endpoints of the intervals \( R_{a_{n-1}}^n (\Delta^{(n)}) \), \( R_{a_{n-1}+q_n}^n (\Delta^{(n)}) \), \( R_{a_{n-1}+2q_n}^n (\Delta^{(n)}) \), \ldots, \( R_{a_{n-1}+(k_{n+1}-1)q_n}^n (\Delta^{(n)}) \)
are of the form \( \alpha \cdot (q_{n-1} + j q_n) \mod 1 \),

\[ 0 \leq j \leq k_{n+1} \]

Notice that the sequence

\[ 9_{-1}, 9_{-1} + 9_0, 9_{-1} + 2q_0, 9_{-1} + 3q_0, \ldots, 9_{-1} + k_1 9_0 = 9_1, \]

\[ 9_0 + q_1, 9_0 + 2q_1, 9_0 + 3q_1, \ldots, 9_0 + k_2 q_1 = 9_2, 9_1 + q_2, \]

\[ 9_1 + 2q_2, 9_1 + 3q_2, \ldots, 9_1 + k_3 q_2 = 9_3, 9_2 + q_3, 9_2 + 2q_3, \ldots \]
is a subsequence of $g_k n \beta n_{z-1}$.
Recall the sequence \( (\mathcal{J}_e^{(n-1)}, \mathcal{J}_e^{(n)}, \mathcal{J}_e^{(n+1)}) \) associated to a via the coding \( J_e^{(n)} \).

What is the map \( F : (0,1) \to (0,1) \) such that

\[
\begin{align*}
\mathcal{J}_e^{(1)}(F(x)) &= \mathcal{J}_e^{(2)}(x) \\
\mathcal{J}_e^{(2)}(F(x)) &= \mathcal{J}_e^{(3)}(x) \\
\mathcal{J}_e^{(3)}(F(x)) &= \mathcal{J}_e^{(4)}(x) \\
\vdots
\end{align*}
\]

i.e. \( F \) acts as a shift \( (\mathcal{J}_e^{(n)}, \mathcal{J}_e^{(n+1)}) \to \mathcal{J}_e^{(n+2)} \)?

**Claim**

\[
F(x) = \begin{cases} 
\frac{1}{1-x} & \text{if } x < \frac{1}{2} \\
2 - \frac{1}{x} & \text{if } x > \frac{1}{2} 
\end{cases}
\]
If, instead of the binary coding \( (f_0(x), f_2(x), \ldots) \), we use the coding \( (\tilde{f}_1(x), \tilde{f}_2(x), \ldots) \), the corresponding map \( \tilde{F} \) is

\[
\tilde{F}(x) = \begin{cases} 
\frac{x}{1-x} & x < \frac{1}{2} \\
\frac{1-x}{x} & x > \frac{1}{2}
\end{cases}
\]

which is usually called the Farey map.
Since the codings $(f_1(a), f_2(a), \ldots)$ or $(\tilde{f}_1(a), \tilde{f}_2(a), \ldots)$ are arbitrary, we can choose to work with $F$ or $\tilde{F}$.

It is usually more convenient to work with $F$.

The Farey fractions can be written as pairs, via $F$ or $\tilde{F}$:

\[
\begin{align*}
\frac{1}{2} & : \frac{2}{3} \\
\frac{1}{3} & : \frac{2}{2} \\
\frac{2}{3} & : \frac{3}{3} \\
\frac{1}{4} & : \frac{2}{5} \\
\frac{3}{5} & : \frac{3}{4}
\end{align*}
\]