Math 347 Honors, Spring 2009, Problem Solutions

HW# 1

1.4 *The square has the largest area among all rectangles with a given perimeter.* The problem may be rewritten as follows: " Given $p > 0$, prove that among all rectangles with perimeter p, the square of side $p/4$ is the one with the largest area." Proof (without calculus): Let p be a positive real number, and let R be a rectangle with perimeter p. If R has side-lengths x, y, then $x + y = p/2$ and the area of R is $xy = x(p/2 - x)$. So the problem is to determine what x gives the maximum of $x(p/2 - x)$. This can be done in an elementary way since $x(p/2 - x)$ has a parabolic shape, or with calculus, or with the AGM inequality (see 1.28 (i)). Either way, the maximum occurs at $x = p/4$. This corresponds to $y = p/4$, in which case R is a square.

Alternatively, if a rectangle has perimeter p, then the sides have lengths $p/4 - y$ and $p/4 + y$ for some y satisfying $0 \le y \le p/4$. The area is then $(p/4 + y)(p/4 - y) = p^2/16 - y^2 \le p^2/16$, with equality only if $y = 0$, i.e., the rectangle is a square.

1.7 *Correction of "If x and y are nonzero real numbers and $x > y$, then $(-1/x) > (-1/y)$."* Analysis: The statement is false when y is negative and x is positive, since $-1/x$ is negative and $-1/y$ is positive, so $(-1/x) < (-1/y)$.

Corrected Statement: "If x and y are nonzero real numbers, $x > y$ and x and y have the same sign, then $(-1/x) > (-1/y)$." Proof: If $x > y$ and x, y have the same signs, then dividing by $(-xy)$, which is a negative number, gives $(-1/y) < (-1/x)$.

It is also correct to modify the statement to "If x and y are nonzero real numbers and $x > y > 0$, then $(-1/x) > (-1/y)$."

1.20 *Roots of a quadratic.* If r and s are the roots of the quadratic $ax^2+bx+c = 0$, then $ax^2+bx+c = a(x-r)(x-s)$ (i.e., the **polynomials** are the same). Expanding the right side gives $ax^2+bx+c = ax^2 - a(r+s)x + ars$. Therefore $r + s = -b/a$ and $rs = c/a$.

1.27 *Determine the set of real solutions to $|x/(x + 1)| \le 1$.* The solutions set is $S = \{x \in \mathbb{R} : x \ge -\frac{1}{2}\}$. Let $T = \{x \in \mathbb{R} : |x/(x+1)| \le 1\}$. If $x \in T$, multiplying by $|x+1|$ then squaring gives $x^2 \le (x+1)^2 = x^2+2x+1$. This is equivalent to $x \ge -\frac{1}{2}$, and it follows that $x \in S$. This proves $T \subseteq S$. Conversely, if $x \in S$, then $2x + 1 \ge 0$. Adding x^2 to both sides gives $(x + 1)^2 \ge x^2$. Now for any $y \in \mathbb{R}$, $\sqrt{y^2} = |y|$. So taking square roots in $(x + 1)^2 \ge x^2$ gives $|x + 1| \ge |x|$. Since $x \ne -1$, we can divide by $|x + 1|$, giving $1 \ge |x/(x + 1)|$, thus $x \in T$. Therefore $S \subseteq T$ and we conclude that $S = T$.

1.28 *Applications of the AGM inequality.* (i) Let c and x be real numbers. By the AGM inequality,

$$x(c - x) \le \left(\frac{x + c - x}{2}\right)^2 = c^2/4.$$

However, when $x = c/2$, $x(c - x) = c^2/4$, so $x(c - x)$ is maximized at $x = c/2$.

(ii) If $a > 0$, then $y(c - ay) = ay(c/a - y)$. By part (i), $y(c/a - y)$ is maximized at $y = \frac{c}{2a}$ and therefore so is $y(c - ay)$.

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HW# 2

1.15 *For sets A and B, $A - B = B - A$ if and only if $A = B$.* If $A = B$, then $A - B = B - A = \emptyset$. Conversely, if $A - B = B - A$, then each element of $A - B$ is not in B (from the definition of $A - B$) and in B (from the definition of $B - A$). This means that $A - B = \emptyset$. Similarly, every element of $B - A$ belongs to B and is not in B, so $B - A = \emptyset$. Lastly, $A - B = \emptyset$ means that $B \subseteq A$, and $B - A = \emptyset$ means that $A \subseteq B$. Therefore, $A = B$.

1.38 *Find a set equality relating $S = \{x \in \mathbb{R}: x(x-1)(x-2)(x-3) < 0\}$, $T = (0,1)$, and $U = (2,3)$.* The relation is $S = T \cup U$. To prove it, we must prove $S \subseteq T \cup U$ and $T \cup U \subseteq S$. First, let $x \in S$. Then $x(x-1)(x-2)(x-3) < 0$, so either one or three of the factors are negative. In the first case, the negative factor must be the smallest one, $x - 3$. Thus $x - 2 > 0$ and $x - 3 < 0$. In other words, $2 < x < 3$, which means that $x \in U$. In the second case, exactly one of the factors is positive and it must be the largest factor, x. Thus, $x > 0$ and $x - 1 < 0$, i.e., $0 < x < 1$. This means $x \in T$. In all cases, $x \in T \cup U$, so $S \subseteq T \cup U$. Next, suppose $x \in T \cup U$. Then either $x \in T$ or $x \in U$. If $x \in T$ then $0 < x < 1$ and so $x(x-1)(x-2)(x-3)$ is the product of one positive and three negative numbers, so it is negative. If $x \in U$, then $2 < x < 3$ and so $x(x-1)(x-2)(x-3)$ is the product of three positive and one negative numbers, so it is negative. In all cases, $x \in S$ and this proves that $T \cup U \subseteq S$. Therefore, $S = T \cup U$.

1.39 *Solutions of* $(x - a_1)(x - a_2)\cdots(x - a_n) < 0$. A product of n real numbers is negative if and only if an odd number of the factors are negative, and none of the numbers is zero. If $x = a_j$ for some number j, then the product is zero. If $x > a_n$, all factors are positive so the product is positive. If $a_{n-1} < x < a_n$, then one factor is negative and the rest are positive, so the product is negative. Similarly, if $a_{n-2} < x < a_{n-1}$, two factors are negative so the product is positive, and if $a_{n-3} < x < a_{n-2}$, then three factors are negative and the product is negative. In general, if $a_{n-2k} < x < a_{n-2k+1}$ for a non-negative integer k, then the product is positive and if $a_{n-2k-1} < x < a_{n-2k}$ for some non-negative integer k, then the product is negative. Finally, when $x < a_1$, the product is negative if n is odd and positive when n is even. The complete solution set is therefore

$$(-\infty, a_1) \cup (a_2, a_3) \cup (a_4, a_5) \cup \cdots \cup (a_{n-1}, a_n)$$

if n is odd, and

$$(a_1, a_2) \cup (a_3, a_4) \cup \cdots \cup (a_{n-1}, a_n)$$

if n is even.

1.41(d) Suppose $A \subseteq B$ and $B \subseteq C$. If $a \in A$, then $a \in B$ by the first assertion and hence $a \in C$ by the second assertion. Thus, for all $a \in A$, $a \in C$. This proves $A \subseteq C$.

1.47(a) *Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by $f(a,b) = (a+1)(a+2b)/2$. (a) Prove that the image of f is contained in $\mathbb{N}$.* Rewrite the function as $f(a,b) = a(a+1)/2 + b(a+1)$. First, $b(a+1)$ is clearly in $\mathbb{N}$ for $a, b \in \mathbb{N}$. Next, $a(a+1)$ is always even because either a or $a+1$ is even for any a, hence $a(a+1)/2 \in \mathbb{N}$. Therefore, $f(a,b) \in \mathbb{N}$.

1.49 (b,c,d,e) *Properties of functions f, g: $\mathbb{R} \to \mathbb{R}$.*

(b) If f and g are bounded, then fg is bounded—TRUE. Since f and g are bounded, there are numbers M_1 and M_2 so that for all x, $|f(x)| \le M_1$ and $|g(x)| \le M_2$. Then, given any x,

$$|f(x)g(x)| = |f(x)||g(x)| \le M_1 M_2.$$

This proves that fg is bounded.

(c) If $f+g$ is bounded, then f and g are bounded—FALSE. The functions f, g defined by $f(x) = x$ and $g(x) = -x$ provide a counterexample. Here f and g are unbounded, but $f(x) + g(x) = 0$ for all x.

(d) If fg is bounded, then f and g are bounded—FALSE. A counterexample is given by $f(x) = x$ for $x > 0$, $f(x) = 0$ for $x \leq 0$, and $g(x) = x$ for all x.

(e) If both fg and $f+g$ are bounded, then f and g are bounded—TRUE. Suppose P and S are positive real numbers such that $|f(x)g(x)| \leq P$ for all x, and $|f(x) + g(x)| \leq S$ for all x. Let $M = P + S + 1$. We claim that $|f(x)| \leq M$ for all x, and $|g(x)| \leq M$ for all x. Suppose that $|f(x)| > P + S + 1$. By the triangle inequality,

$$P + S + 1 < |f(x)| = |f(x) + g(x) - g(x)| \leq |f(x) + g(x)| + |g(x)|,$$

which implies that

$$|g(x)| \geq P + S + 1 - |f(x) + g(x)| \geq P + 1.$$

Therefore,

$$|f(x)g(x)| = |f(x)| \cdot |g(x)| \geq (P + S + 1)(P + 1) > P,$$

which is a contradiction. Therefore, $|f(x)| \leq P + S + 1$ for all x. The proof that $|g(x)| \leq P + S + 1$ for all x is similar.

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HW# 3

2.4 *Negation of sentences, where $A, B \subseteq \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$, and $P = \{x \in \mathbb{R}\colon x > 0\}$.*

(a) For all $x \in A$, there is a $b \in B$ such that $b > x$. *Negation:* There is an $x \in A$, such that for all $b \in B$, $b \leq x$.

(b) There is an $x \in A$ such that, for all $b \in B$, $b > x$. *Negation:* For all $x \in A$, there is some $b \in B$ such that $b \leq x$.

(c) For all $x, y \in \mathbb{R}$, $f(x) = f(y) \Rightarrow x = y$. *Negation:* There are numbers $x, y \in \mathbb{R}$ such that $f(x) = f(y)$ and $x \neq y$.

(d) For all $b \in \mathbb{R}$, there is an $x \in \mathbb{R}$ such that $f(x) = b$. *Negation:* There is some $b \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x) \neq b$.

(e) For all $x, y \in \mathbb{R}$ and all $\epsilon \in P$, there is a $\delta \in P$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. *Negation:* There are $x, y \in \mathbb{R}$ and $\epsilon \in P$ such that for every $\delta \in P$, $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

(f) For all $\epsilon \in P$, there is a $\delta \in P$ such that, for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. *Negation:* There is an $\epsilon \in P$ such that, for all $\delta \in P$, there are $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

2.10 *More negations.*

(a) Every odd number is prime. *Logical form*: For all odd numbers n, n is prime. *Negation:* There is an odd number which is not prime.

(b) The sum of the angles of a triangle is 180 degrees. *Logical form:* For every triangle T, the sum of the angles of T is 180 degrees. *Negation:* There is a triangle T such that the sum of the angles of T is not 180 degrees.

(c) Passing the test requires solving all the problems. *Logical form:* If you pass the test, then you solved all the problems. *Negation:* You passed the test, but did not solve all the problems.

(d) Being first in line guarantees getting a good seat. *Logical form*: If you are first in line, the you will get a good seat. *Negation:* You are first in line and do not get a good seat.

(e) Lockers must be turned in by the last day of class. *Logical form:* For all lockers, the locker must be turned in by the last day of class. *Negation*: There is some locker which does not have to be turned in by the last day of class.

OR logical form: If you have a locker, then it must be turned in by the last day of class. *Negation:* You have a locker, and it doesn't have to be turned in by the last day of class.

2.21 *Negation of "For every integer $n > 0$, there is some real number $x > 0$ such that $x < 1/n$."*. There is an integer $n > 0$ such that for every real number $x > 0$, $x \geq 1/n$. The original sentence is true: Given an integer $n > 0$, take $x = \frac{1}{n+1}$.

2.22 *Meaning of "f is not an increasing function"*. The meaning of "f is an increasing function" is: For all $x, y \in \mathbb{R}$, if $x > y$, then $f(x) > f(y)$. We want the negation of this: There are $x, y \in \mathbb{R}$ so that $x > y$ and $f(x) \leq f(y)$.

2.23 *The meaning of "$g \notin S$", where* $S = \{g\colon \mathbb{R} \to \mathbb{R}\colon (\exists c, a \in \mathbb{R})(x > a \Rightarrow |g(x)| \leq c|f(x)|)\}$. Note that S depends on f. The phrase "$g \notin S$" is the negation of $g \in S$. It means: For all $c, a \in \mathbb{R}$, there is a real number $x > a$ such that $|g(x)| > c|f(x)|$.

2.24 *Two statements about a set S of real numbers.*

(a) There is a number M such that for every $x \in S$, $|x| \leq M$.

(b) For every $x \in S$, there is a number M such that $|x| \leq M$.

Solution: (a) says that S is bounded and M is a bound for the entire set S (Definition 1.31). Its negation is "For all numbers M, there is an $x \in S$ such that $|x| > M$." This means that S is unbounded.

(b) says that each element x of S has a bound M, but the numbers M can be different for different x. Thus, (b) actually says *nothing* about the set S, and is always true. Its negation is "There is some $x \in S$ such that for every M, $|x| > M$", which is always false.

Since (a) is not always true (e.g., $S = \mathbb{R}$), the correct implication is $(a) \implies (b)$. Alternatively, you can argue that if (a) is true, then there is some M that "works" for all $x \in S$ simultaneously. In particular, for all $x \in S$, there is some M that works for that x, which is (b).

2.31 *Which statements are believable?*

(a) "All my 5-legged dogs can fly". True, because I have no 5-legged dogs.

(b) "I have no 5-legged dog that cannot fly". This is logically equivalent to (a), so is also True.

(c) "Some of my 5-legged dogs cannot fly". False, since I have no 5-legged dogs at all.

(d) "I have a 5-legged dog than cannot fly". Logically equivalent to (c), so also False.

2.32 *Fraternity pledges.* Each person always tells the truth or always lies:
A) All three of us are liars.
B) Only two of us are liars.
C) The other two are liars.

Suppose the statement of A is true. Then it must be false, which is a contradiction. Hence it must be false and A is a liar. If the statement of C is true, B is a liar. But then A,B are liars and C tells the truth, which makes the statement of B true. This is also a contradiction, therefore C is a liar. Now we know A and C are liars. Since A's statement is false, B can't also be a liar, so B tells the truth.

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HW# 4

2.19 *Quote of Abraham Lincoln.* Logically, his statement means: "There is a time t such that for all persons x, x is fooled at time t, AND there exists a person x such that for all times t, x is fooled at time t, AND it is not true that for all persons x and for all time, x is fooled at time t." There are three claims made in this statement. The negation would be that at least one of them is false: *For all time t, there is a person x who is not fooled at time t, OR for all persons x, there is some time t when x is not fooled, OR for all persons x and for all time t, x is fooled at time t."* Which is true is not something that can't be proved, although my feeling is that his original statement is true.

2.38 *Parity of products of integers x, y.*

(a) xy is odd if and only if x and y are both odd. – TRUE. If x and y are odd, then $x = 2k+1$ and $y = 2l+1$ for some integers k, l. Then

$$xy = (2k+1)(2l+1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1,$$

so xy is odd. To prove the implication "if xy is odd, then x and y are odd", we prove the contrapositive: "if x is even or y is even, then xy is even". If x or y is even, then one of x, y is a multiple of 2, and therefore xy is also a multiple of 2.

Alternately, one can look at the four possibilities for the parities of x and y, and show that in all four cases, "xy is odd" and "x and y are both odd" are either both true or both false.

(b) xy is even if any only if x and y are both even. – FALSE. For example, if $x = 1$ and $y = 2$, then $xy = 2$.

2.47 *Quantifiers and conditional statements.* Let $P(x)$ be "x is odd", and let $Q(x)$ be "$x^2 - 1$ is divisible by 8".

a) $(\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x))$*—TRUE.* Consider an integer x. Under the hypothesis "x is odd", we have $x = 2k+1$ for some integer k, and hence $x^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k+1)$. When k is an integer, one of k and $k+1$ is even, and hence $k(k+1)$ is even. Therefore, $4k(k+1)$ is divisible by 8.

b) $(\forall x \in \mathbb{Z})(Q(x) \Rightarrow P(x))$*—TRUE.* For (b), we prove for each x the contrapositive $\neg P(x) \Rightarrow \neg Q(x)$. If x is not odd, then x is even, so x^2 is even, so $x^2 - 1$ is odd and hence not divisible by 8.

3.5 *For $n \in \mathbb{N}$, $\sum_{k=1}^{n}(2k+1) = n^2 + 2n$—TRUE.* For $n = 1$, $2 \cdot 1 + 1 = 3 = 1^2 + 2 \cdot 1$. If $P(n)$ is true, i.e. $\sum_{k=1}^{n}(2k+1) = n^2 + 2n$, then by substitution

$$\sum_{k=1}^{n+1}(2k+1) = \sum_{k=1}^{n}(2k+1) + 2(n+1) + 1 = (n^2+2n) + (2(n+1)+1) = (n+1)^2 + 2(n+1).$$

So $P(n+1)$ is true. Therefore by induction $\sum_{k=1}^{n}(2k+1) = n^2 + 2n$ for all $n \in \mathbb{N}$.

3.16 *For $n \in \mathbb{N}$, $\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$.* We use induction on n. For $n = 1$, we have $\sum_{i=1}^{1} i^3 = 1 = (\frac{1 \cdot 2}{2})^2$, which completes the basis step.

For the induction step, suppose that the claim holds when $n = k$. Using the induction hypothesis after isolating the last term, we have

$$\sum_{i=1}^{k+1} i^3 = (k+1)^3 + \sum_{i=1}^{k} i^3 = (k+1)^3 + \left(\frac{k(k+1)}{2}\right)^2$$
$$= \frac{(k+1)^2}{4}[4(k+1) + k^2] = \frac{(k+1)^2}{4}(k+2)^2.$$

Hence the claim holds also when $n = k + 1$. By induction, the statement is true for all $n \in \mathbb{N}$.

3.18 If $0 \leq a_i \leq b_i$ for $i \in \mathbb{N}$, $\prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} b_i$. Proof: Since $a_1 \leq b_1$ is given, the inequality is true for $n = 1$. Assume the inequality is true for $n = k$. By the given information $0 \leq a_{k+1} \leq b_{k+1}$, it follows that

$$\prod_{i=1}^{k+1} a_i = a_{k+1} \prod_{i=1}^{k} a_i \leq a_{k+1} \prod_{i=1}^{k} b_i \leq b_{k+1} \prod_{i=1}^{k} b_i.$$

Thus the inequality is true for $n = k + 1$. By induction, the inequality is true for all $n \in \mathbb{N}$.

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HW# 5

3.4 *If $P(0)$ is true, and the truth of $P(n)$ implies the truth of $P(n+1)$ or $P(n-1)$, then possibly only two of the indexed statements are true.* Since $P(0)$ is true, $P(1)$ or $P(-1)$ must be true. However, the truth of $P(0)$ and $P(1)$ does not imply that any other statements are true, and neither does the truth of $P(0)$ and $P(-1)$. Therefore, the only statement we know the truth value of is $P(0)$. We also know at least one of $P(1), P(-1)$ is true. We know absolutely nothing about $P(n)$ for other n.

3.24 *(1) $m \in T$ and (2) $n \in T \implies n+1 \in T$. Is $T = \{n \in \mathbb{N} : n \geq m\}$?* Using (1) as a base step and (2) as an induction step proves that all integers $n \geq m$ lie in T. However, we do not know whether or not any integers $n < m$ lies in T. So we cannot conclude that $T = \{n \in \mathbb{N} : n \geq m\}$. We can only conclude that $T \supseteq \{n \in \mathbb{Z} : n \geq m\}$. For example, if $T = \mathbb{N}$ and $m = 4$, then (1) and (2) are true, but $T \neq \{n \in \mathbb{N} : n \geq 4\}$.

3.31 *For $n \in \mathbb{N}$ and $n \geq 2$, $\prod_{i=2}^{n}(1 - \frac{1}{i^2}) = \frac{n+1}{2n}$.* The first few values are $\frac{3}{4}, \frac{4}{6}, \frac{5}{8}$; the pattern suggests the formula $\frac{n+1}{2n}$, which we prove by induction on n. The key observation is that $1 - \frac{1}{i^2} = \frac{(i-1)(i+1)}{i \cdot i}$. For $n = 2$, $1 - 1/4 = 3/4 = \frac{3}{2 \cdot 2}$, as desired. For the induction step, suppose that the claim holds when $n = k$. Using the induction hypothesis for $n = k$, we have

$$\prod_{i=2}^{k+1}\left(1 - \frac{1}{i^2}\right) = \frac{k+1}{2k}\left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+1}{2k}\,\frac{k(k+2)}{(k+1)\cdot(k+1)} = \frac{k+2}{2(k+1)}.$$

Hence the claim also holds when n is $k+1$. By induction, the formula holds for all $n \geq 2$.

3.41 *A function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for $x, y \in \mathbb{R}$.*

a) $f(0) = 0$. Taking $x = y = 0$, we obtain $f(0+0) = f(0) + f(0)$. Thus $f(0) = 2f(0)$, which requires $f(0) = 0$.

b) $f(n) = nf(1)$ for $n \in \mathbb{N}$. We use induction on n. Since $f(1) = 1 \cdot f(1)$, the claim holds when $n = 1$. For the induction step, suppose that $f(n-1) = (n-1)f(1)$. Since $f(n-1+1) = f(n-1) + f(1) = (n-1)f(1) + f(1) = nf(1)$, the claim holds also at the next value.

3.49b *determine the $n \in \mathbb{N}$ so that $2^n \geq (n+1)^2$.* The inequality holds for all $n \geq 6$. By direct calculation, the inequality fails for $1 \leq n \leq 5$. Let $P(n)$ be the statement $2^n \geq (n+1)^2$. $P(6)$ is true because $64 \geq 49$. Assume $k \geq 6$ and $P(k)$ is true, that is, $2^k \geq (k+1)^2$. Doubling both sides of the inequality yields $2^{k+1} \geq 2(k+1)^2$. We want the right side to be $\geq (k+2)^2$. Since $k \geq 6$, $k^2 \geq 36$, so

$$2^{k+1} \geq 2(k+1)^2 = k^2 + 4k + 2 + k^2 \geq k^2 + 4k + 4 = (k+2)^2,$$

which proves $P(k+1)$. By induction, $P(k)$ is true for all $k \geq 6$.

3.56 $a_n = 2a_{n-1} + 3a_{n-2}$ *for $n \geq 2$.* (a) *If a_1 and a_2 are odd, then a_n is odd for all $n \in \mathbb{N}$.* Proof by induction on n. For the base case, we are given that a_1 and a_2 are odd. Suppose $k \geq 2$ and assume $a_1, \ldots, a_k$ are all odd. In particular, a_k and a_{k-1} are odd. By the recurrence formula, $a_{k+1} = 2a_k + 3a_{k-1}$. Since a_k is odd, $2a_k$ is even. Since a_{k-1} is odd, $3a_{k-1}$ is odd. An even number plus an odd number is odd, so a_{k+1} is odd. By induction, a_n is odd for all $n \in \mathbb{N}$.

(b) The formula holds for $n = 1$ and $n = 2$, and these two cases form the base step. Assume that $k \geq 2$ and that the formula is true for $1 \leq n \leq k$. In particular, the formula is true for $n = k$ and $n = k-1$. Then

$$a_{k+1} = 2a_k + 3a_{k-1} = 2\frac{1}{2}(3^{k-1} - (-1)^k) + 3\frac{1}{2}(3^{k-2} - (-1)^{k-1}) = \frac{1}{2}(3^k - (-1)^{k+1}),$$

since $2(-1)^k + 3(-1)^{k-1} = (-1)^{k-1}(-2+3) = (-1)^{k-1} = (-1)^{k+1}$. Thus, the formula is true for $n = k$. By induction, the formula is true for all $n \in \mathbb{N}$.

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HW# 6

4.11 *Multiplication by 2 defines a bijection from* $\mathbb{R}$ *to* $\mathbb{R}$ *but not from* $\mathbb{Z}$ *to* $\mathbb{Z}$. Let f denote the doubling function. For $y \in \mathbb{R}$, the number $x = y/2$ is the unique real number such that $f(x) = y$. When $y \in \mathbb{Z}$ and y is odd, $y/2 \notin \mathbb{Z}$. Hence odd numbers are not in the image of $f\colon \mathbb{Z} \to \mathbb{Z}$, so f is not a bijection from $\mathbb{Z}$ to $\mathbb{Z}$.

4.12 *Properties of functions.*

a) Every decreasing function from $\mathbb{R}$ *to* $\mathbb{R}$ *is surjective—FALSE.* Take $f(x) = e^{-x}$.

b) Every nondecreasing function from $\mathbb{R}$ *to* $\mathbb{R}$ *is injective—FALSE.* The constant function f defined by $f(x) = 0$ is nondecreasing but not injective.

c) Every injective function from $\mathbb{R}$ *to* $\mathbb{R}$ *is monotone—FALSE.* Take $f(x) = x$ for $x \leq 0$ and $f(x) = 1/x$ for $x > 0$. Then f is injective, but not monotone.

4.24 *If* f *and* g *are surjections from* $\mathbb{Z}$ *to* $\mathbb{Z}$, *must* $h = fg$ *be surjective?* NO. For example, take $f(n) = n$ and $g(n) = n$ for all $n \in \mathbb{Z}$. then $h(n) = n^2$, which is not a surjective function from $\mathbb{Z}$ to $\mathbb{Z}$.

4.25 *Formulas defining surjections from* $\mathbb{N} \times \mathbb{N}$ *to* $\mathbb{N}$.

c) $f(a,b) = ab(b+1)/2$*—YES.* For $n \in \mathbb{N}$, $f(n,1) = n$, so n is in the image.

d) $f(a,b) = (a+1)b(b+1)/2$*—NO.* When $a, b \in \mathbb{N}$, $(a+1)b(b+1)/2 \geq 2$, so the image does not contain 1.

e) $f(a,b) = ab(a+b)/2$*—NO.* We have $f(1,1) = 1$. When $\min\{a,b\} = 1$ and $\max\{a,b\} \geq 2$, we have $ab(a+b)/2 \geq 3$. When $a, b \geq 2$, we have $ab(a+b)/2 \geq 8$. Thus the image does not contain 2.

4.26 *If there are positive constants* c, α *such that, for all* $x, y \in \mathbb{R}$, $|f(x) - f(y)| \geq c\,|x-y|^\alpha$, *then* f *is injective.* If f is not injective, then there are distinct numbers x, y such that $f(x) = f(y)$. Since $c\,|x-y|^\alpha > 0$, this contradicts the hypothesized condition.

4.29 (a) Since $g(1) = g(-1)$, g is not injective. Since $f(2) = f(1/2)$, f is not injective. In fact, $f(x) = f(y)$ if and only if $xy = 1$. Lastly, $h'(x) = \frac{x^4+3x^2}{(1+x^2)^2}$, which is > 0 except at $x = 0$. Thus, h is an increasing function and is therefore injective.

(b) Obviously g cannot take negative values, so g is not surjective. For f, note that when $x < 0$, $f(x) < 0$. When $0 < x < 1$, $0 < f(x) < 1$ and when $x \geq 1$, $f(x) \leq \frac{x^2}{1+x^2} \leq 1$. Thus, $f(x) \leq 1$ for all x, therefore $f(x) = 2$ has no solutions. Hence f is not surjective.

4.31 If $A, B \subseteq \mathbb{R}$, $f : A \to B$ is a bijection and increasing, then f^{-1} is increasing. We prove the contrapositive: if f^{-1} is not increasing, then f is not increasing. If f^{-1} is not increasing, this means that for some numbers $b_1, b_2 \in B$ we have $b_1 < b_2$ and $f^{-1}(b_1) \geq f^{-1}(b_2)$. It is not possible that $f^{-1}(b_1) = f^{-1}(b_2)$ because f^{-1} is a bijection, therefore $f^{-1}(b_1) > f^{-1}(b_2)$. Let $a_1 = f^{-1}(b_1)$ and $a_2 = f^{-1}(b_2)$. Then $a_1 > a_2$ and $f(a_1) < f(a_2)$, which proves that f is not increasing.

4.34 *Properties of* $h = g \circ f$, *where* $f : A \to B$, $g : B \to C$.

(a) *If* h *is injective, then* f *is injective.* True. Suppose h is injective and f is not injective. Then there are different elements $a_1 \in A$, $a_2 \in A$ such that $f(a_1) = f(a_2) = b$, where $b \in B$. Then $h(a_1) = g(f(a_1)) = g(b) = g(f(a_2)) = h(a_2)$, contradicting the injectivity of f.

(b) *If* h *is injective, then* g *is injective.* False. Let $A = [0, \infty)$, $B = C = \mathbb{R}$, $f(x) = x$, $g(x) = x^2$.

(c) *If* h *is surjective, then* f *is surjective.* False. Let $A = B = \mathbb{R}$, $C = [0, \infty)$, $f(x) = x^2$, $g(x) = x^2$.

(d) *If* h *is surjective, then* g *is surjective.* True. Since h is surjective, for every $c \in C$ there is an $a \in A$ with $h(a) = c$. But then $g(f(a)) = c$, so g is surjective also.

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HW# 7

5.5 There are n was to choose the woman, and for each choice, there are $n-1$ ways to choose the man who is not married to her. A total of $n(n-1)$ ways.

5.10 There are 2^{2n} strings of H's and T's. We choose n positions among $2n$ possible positions for the heads, for a total of $\binom{2n}{n}$ ways. The probability is therefore $\binom{2n}{n}/2^{2n}$. When $n = 10$, the probability is

$$2^{-20}\binom{20}{10} = \frac{46189}{262144}$$

in reduced terms.

5.12 There are $6^3 = 216$ possibilities (6 possibilities for each roll). The ways to reach a sum of 11 are $(1, 4, 6)$ and 5 other permutations of these numbers, same for $(2, 3, 6)$ and $(2, 4, 5)$. Also $(1, 5, 5)$, $(3, 3, 5)$ and $(3, 4, 4)$, each with 2 other rearrangements. The total ways is thus $6 + 6 + 6 + 3 + 3 + 3 = 27$. The probability is $27/216 = 1/8$.

5.19 *Counting 6-digit numbers with 2 or 3 distinct digits.* There are $\binom{10}{2}$ ways to choose the two digits, and $2^6 - 2$ ways to choose the sequence of 6 of these digits (the two sequences of all the same digit are not allowed), for a total of $\binom{10}{2}(2^6 - 2) = 2790$ numbers.

When $k = 3$, choose the 3 distinct digits ($\binom{10}{3}$ ways), then choose a 6-tuple consisting of these digits. The second part is done by taking all possible 6-tuples of the 3 numbers (3^6 of them), and subtracting off the ones that use only 1 or two of them. There are $3(2^6 - 2)$ strings using only two of the three digits (3 choices for the two numbers, $2^6 - 2$ choices for the string by the $k = 2$ case), and 3 strings with only one digit used. Thus, the number of numbers with 3 distinct digits is $\binom{10}{3}(3^6 - 3(2^6 - 2) - 3) = 64800$.

5.28 *Solutions of $x_1 + \cdots + x_k \le n$ in non-negative integers.*

Proof 1. For each $m \ge 0$, the number of solutions of $x_1+\cdots+x_k = m$ is $\binom{m+k-1}{k-1}$. Summing over $0 \le m \le n$ by Theorem 5.28 gives a total of $\binom{n+k}{k}$ solutions.

Proof 2. Let $x_{k+1} = n - (x_1 + \cdots + x_k)$, so that $x_1 + \cdots + x_k + x_{k+1} = n$. This equation has exactly the same number of solutions as the original inequality, which is $\binom{k+1+n-1}{k+1-1} = \binom{n+k}{k}$.

5.42 *Prove that* $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$. Let A be an m-element set, B a disjoint $n-$element set, and $C = A \cup B$, so $|C| = m + n$. The right side is the number of ways to choose k elements from C. In doing so, i of these elements are from A and $k - i$ elements from B, for some i satisfying $0 \le i \le k$. For each i, the number of choices is $\binom{m}{i}\binom{n}{k-i}$. Summing over i gives the total number of choices.

5.45 *show that* $\sum_{A\subseteq[n]} \sum_{B\subseteq[n]} |A \cap B| = n4^{n-1}$. The left side counts triples (x, A, B) where $1 \le x \le n$, $A \subseteq [n]$, $B \subseteq [n]$, and $x \in A \cap B$. For each x, there are 2^{n-1} subsets of A containing x, and there are 2^{n-1} subsets of B containing x. Hence, for each x, there are 4^{n-1} pairs A, B with $x \in A \cap B$. Hence the left side of the equation is $n4^{n-1}$.

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HW# 8

6.4 *Suppose* $\gcd(a, b) = 1$. *Prove* $\gcd(na, nb) = n$. Clearly n divides both na and nb, hence $\gcd(na, nb) \geq n$. By Lemma 6.5, there are integers x, y so that $ax + by = 1$. Multiplying through by n, we find that $an(x) + bn(y) = n$, so n is an integer combination of an and bn. By Theorem 6.12, n is a multiple of $\gcd(na, nb)$, i.e. $\gcd(na, nb) \leq n$. Therefore, $\gcd(na, nb) = n$.

6.17 Using Proposition 6.13 with $k = 1$, $\gcd(a+b, a-b) = \gcd(a+b+(a-b), a-b) = \gcd(2a, a-b)$ and

$\gcd(a+b, a-b) = \gcd(a-b, a+b) = \gcd(a-b-(a+b), a+b) = \gcd(-2b, a+b) = \gcd(2b, a+b) = \gcd(a+b, 2b)$.

6.18 Suppose $a = p_1 \cdots p_k$ and $b = q_1 \cdots q_m$, where each p_i and q_i is a prime number. Since $\gcd(a, b) = 1$, no p_i equals any q_j. First write $a^2 = p_1^2 \cdots p_k^2$ and $b^2 = q_1^2 \cdots q_m^2$. This means there is no prime dividing both a^2 and b^2, hence $\gcd(a^2, b^2) = 1$. Next write $2b = 2q_1 \cdots q_m$. If $2|a$, then none of the q_i equal 2, so $\gcd(a, 2b) = 2$. If $2 \nmid a$, $\gcd(a, 2b) = 1$.

6.22 $k-2$ *divides* $2k$ *for positive integers* $k = 3, 4, 6$ *only.* $k-2$ divides $2k$ if and only if $\frac{2k}{k-2}$ is an integer. Since $\frac{2k}{k-2} = 2 + \frac{4}{k-2}$, this occurs if and only if $\frac{4}{k-2}$ is an integer, i.e., if and only if $k-2$ divides 4. The only positive divisors of 4 are 1,2,4, corresponding to $k = 3, 4, 6$.

6.24 (i) When $n = 1$, $3|(4^1 - 1)$. Assume $3|(4^m - 1)$ for some $m \in \mathbb{N}$. Then $4^m - 1 = 3k$ for some $k \in \mathbb{Z}$. This implies that $4^{m+1} - 4 = 12k$, or $4^{m+1} - 1 = 12k + 3 = 3(4k+1)$, thus $3|(4^{m+1} - 1)$. By induction, $3|(4^n - 1)$ for all $n \in \mathbb{N}$.

(ii) When $n = 1$, $6|(1^3 + 5 \cdot 1)$. Assume that $6|(m^3 + 5m)$ for some $m \in \mathbb{N}$. Then $m^3 + 5m = 6k$ for some $k \in \mathbb{Z}$. Then

$$(m+1)^3 + 5(m+1) = m^3 + 3m^2 + 8m + 6 = 6k + 3m^2 + 3m + 6 = 6(k+1) + 3m(m+1).$$

Since m or $m+1$ is even, $m(m+1)$ is even. Thus, $6|3m(m+1)$ and hence $6|((m+1)^3 - 5(m+1))$. By induction, $6|(n^3 + 5n)$ for all $n \in \mathbb{N}$.

6.28 *Suppose* $\gcd(a, b) = 1$ *and that* $a|n$ *and* $b|n$. *Prove that* $ab|n$. Write prime factorizations for a and b as

$$a = p_1^{e_1} p_2^{e^2} \cdots p_k^{e_k}, \qquad b = q_1^{f_1} q_2^{f_2} \cdots q_h^{f_h}.$$

Since $a|n$ and $b|n$, n contains in its own prime factorization $p_1^{e_1}$ (or a higher power of p_1), ..., $p_k^{e_k}$, $q_1^{f_1}$, ..., and $q_h^{f_h}$. Since a and b are relatively prime, $p_i \neq q_j$ for all i, j. Thus n must be divisible by

$$p_1^{e_1} p_2^{e^2} \cdots p_k^{e_k} q_1^{f_1} q_2^{f_2} \cdots q_h^{f_h}.$$

That is, n is divisible by ab.

6.29 *The product of the least common multiple and greatest common divisor of* a *and* b *is* $a \cdot b$. Let $p_1, p_2, \ldots, p_k$ be the primes dividing ab. Suppose a and b have factorizations

$$a = \prod_{i=1}^{k} p_i^{a_i}, \qquad b = \prod_{i=1}^{k} p_i^{b_i}.$$

Here some of the numbers a_i, b_i can be zero. Let $d = \gcd(a, b)$ and $m = \text{lcm}(a, b)$. Since d is the gcd, the exponent of p_i in the prime factorization of d must be $\min\{a_i, b_i\}$, and similarly the exponent of p_i in the prime factorization of m must be $\max\{a_i, b_i\}$. Thus

$$dm = \prod_{i=1}^{k} p_i^{\min\{a_i, b_i\} + \max\{a_i, b_i\}} = \prod_{i=1}^{k} p_i^{a_i + b_i} = ab.$$

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HW# 9

6.31 (b) *Divisibility properties of integer solutions to* $a^2 + b^2 = c^2$*. If* c *is divisible by 3, then* a *and* b *are both divisible by 3.* Consider a solution with c divisible by 3. If a is divisible by 3, a^2 is divisible by 3, and so is $b^2 = c^2 - a^2$. So b is divisible by 3. Likewise, if $3|b$ then $3|a$. Hence both a, b are divisible by 3 or neither is divisible by 3. If x is not divisible by 3, then it can be written as $3k + 1$ or $3k + 2$ for some integer k (in other words, the remainder after dividing by 3 must be 1 or 2). In each case x^2 is 1 more than a multiple of 3: $(3k + 1)^2 = 9k^2 + 6k + 1$ and $(3k + 2)^2 = 9k^2 + 12k + 4$. Hence if neither of a, b is a multiple of 3, then $a^2 + b^2$ is 2 more than a multiple of 3. This is impossible, since we have assumed that their sum is divisible by 3. Hence a, b must both be divisible by 3.

6.40 *If* $2^n - 1$ *is prime, then* $2^{n-1}(2^n - 1)$ *is perfect (equal to the sum of its proper divisors).* Let $q = 2^n - 1$, which is prime. The proper divisors of this number are $1, 2, 2^2, \ldots, 2^{n-1}$ and $q, 2q, 2^2q, \ldots, 2^{n-2}q$. The sum of all these numbers is

$$(q + 1)(1 + 2 + 2^2 + \cdots + 2^{n-2}) + 2^{n-1} = 2^n(2^{n-1} - 1) + 2^{n-1} = 2^{2n-1} - 2^{n-1} = 2^{n-1}(2^n - 1).$$

6.50 (a) Suppose that there are natural numbers m, n so that $\frac{7}{17} = \frac{1}{m} + \frac{1}{n}$. We may assume that $m \leq n$. Then $\frac{1}{m} \leq \frac{7}{17} \leq \frac{2}{m}$, so $3 \leq m \leq 4$. Trying the possibilities $m \in \{3, 4\}$, in each case n would not be an integer. Therefore, m and n do not exist.

7.6 *Remainders modulo 8:* $9^{1000} \equiv 1 \pmod 8$, $10^{1000} \equiv 0 \pmod 8$, $11^{1000} \equiv 1 \pmod 8$. Determining the last digit in the base 8 expansion is the same as finding the congruence class modulo 8. Since $9 \equiv 1 \pmod 8$, 9^{1000} has the same remainder as 1^{1000}. Since $10 = 2 \cdot 5$, 10^{1000} is a multiple of $2^3 = 8$, so its remainder is 0. Since $11 \equiv 3 \pmod 8$, $11^{1000} \equiv (3^2)^{500} \equiv 1^{500} \equiv 1 \pmod 8$.

7.11 *Relations: (a)* $S = \mathbb{N} - \{1\}$*;* $(x, y) \in R$ *if and only if* $\gcd(x, y) > 1$. This satisfies the reflexive and symmetric properties, but not the transitive property. Counterexample: $x = 2$, $y = 6$, $z = 3$. Then $x \sim y$ ($\gcd(x, y) = 2$) and $y \sim z$ ($\gcd(y, z) = 3$), but $x \not\sim z$ ($\gcd(x, z) = 1$). Therefore R is not an equivalence relation. *(b).* $S = \mathbb{R}$*;* $(x, y) \in R$ *if and only if there exists* $n \in \mathbb{Z}$ *such that* $x = 2^n y$. (i) The reflexive property holds because $x = 2^0 x$ for all $x \in \mathbb{R}$. (ii) The symmetric property holds because if $x = 2^n y$ with $n \in \mathbb{Z}$, then $y = 2^{-n} x$. (iii) The transitive property holds: if $x = 2^n y$ and $y = 2^m z$ with $n, m \in \mathbb{Z}$, then $x = 2^{m+n} z$. Therefore R is an equivalence relation.

7.14 Let $f : \mathbb{R} \to \mathbb{R}$. First, for all $g \in S$, and for all x, $g(x) - g(x) = 0 \leq |f(x)|$, so $f \sim f$. If $g, h \in O(f)$, then there are positive real numbers c, a so that for all $x > a$, $|g(x) - h(x)| \leq c|f(x)|$. But this implies $|h(x) - g(x)| \leq c|f(x)|$ for all $x > a$, so $h - g \in O(f)$. Finally, suppose $g - h \in O(f)$ and $h - k \in O(f)$. Then for some positive real numbers a_1, a_2, c_1, c_2, we have

$$\forall x > a_1 : |g(x) - h(x)| \leq c_1|f(x)|, \qquad \forall x > a_2 : |h(x) - k(x)| \leq c_2|f(x)|.$$

Let $a = \max(a_1, a_2)$ and $c = c_1 + c_2$. By the Triangle Inequality, for all $x > a$,

$$|g(x) - k(x)| = |(g(x) - h(x)) + (h(x) - k(x))| \leq |g(x) - h(x)| + |h(x) - k(x)| \leq (c_1 + c_2)|f(x)| = c|f(x)|.$$

Therefore, $g - k \in O(f)$. Since the relation satisfies the reflexive, symmetric and transitive properties, R is n equivalence relation.

7.18 *Solution to* $2n^2+n \equiv 0 \pmod p$ *when* p *is an odd prime.* We factor $2n^2+n$ as $n(2n+1)$ and ask when the product is a multiple of p. Since p is prime, this occurs if and only if $p|n$ or $p|(2n+1)$, i.e. either $n \equiv 0 \pmod p$ or $2n+1 \equiv 0 \pmod p$. The second congruence is $2n \equiv p-1 \pmod p$. Since p is odd, we can divide by 2, yielding $n \equiv (p-1)/2 \pmod p$.

BONUS . If $n > 4$ and n is composite, then $(n-1)! \equiv 0 \pmod n$. Write the prime factorization of n as $n = p_1^{e_1} \cdots p_k^{e_k}$. Then $k \geq 2$ or ($k = 1$ and $e_1 \geq 2$). In the first case, for each j, $p_j^{e_j} < n$, so $(n-1)!$ has a factor $p_j^{e_j}$. Thus $(n-1)!$ is divisible by n. In the second case, if $e_1 \geq 3$, then $p_1, p_1^2, \cdots, p_1^{e_1-1}$ are all factors of $(n-1)!$ and

$$p_1 \cdot p_1^2 \cdots p_1^{e_1-1} = p_1^{e_1(e_1-1)/2}.$$

Since $e_1(e_1-1)/2 \geq e_1$, $(n-1)!$ is divisible by $p_1^{e_1} = n$. Lastly, if $e_1 = 2$, then $p_1 \geq 3$ because $n > 4$. Then p_1 and $2p_1$ are both factors in $(n-1)!$, so again $n = p_1^2$ divides $(n-1)!$.

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HW# 10

7.24 *When is $f : \mathbb{Z}_n \to \mathbb{Z}_n$, $f(x) = x^2$ an injection.* The function is an injection only for $n = 1$ and $n = 2$. When $n \geq 3$, $f(1) = f(n-1) = 1$, so f is not an injection.

7.30 *Primes and threes.* (a) Suppose n has a decimal expansion $n = d_k 10^k + d_{k-1}10^{k-1} + \cdots + d_1 10 + d_0$, where $d_k, \ldots, d_0$ are the digits of n. Since $10 \equiv 1 \pmod 3$, $10^m \equiv 1 \pmod 3$ for all positive integers m. Thus,

$$n \equiv d_k + d_{k-1} + \cdots + d_1 + d_0 \pmod 3.$$

The right side of the congruence is the sum of the digits of n, so clearly that sum is $0 \mod 3$ if and only if $n \equiv 0 \pmod 3$.

(b) Suppose $x - 1$ and $x + 1$ are both prime. Since there is only one even prime, namely 2, x must be even. That is, $2|x$. If x is not divisible by 3, then $x \equiv 1 \pmod 3$ or $x \equiv 2 \pmod 3$. In the first case, $3|(x-1)$, so $x - 1$ won't be prime unless $x - 1 = 3$ (then $x + 1 = 5$ is prime). In the second case, $3|(x + 1)$, which isn't prime unless $x + 1 = 3$ (but then $x - 1 = 1$, which isn't prime). Therefore, with the exception of $x = 4$, $3|x$ if $x - 1$ and $x + 1$ are prime. Since $2|x$ and $3|x$, $6|x$ when $x - 1$ and $x + 1$ are prime, with the one exception already mentioned.

10.4 Let S be a set of $n + 1$ numbers in $\{1, 2, \ldots, 2n\}$.Say $S = \{s_1, s_2, \ldots, s_{n+1}\}$ with $1 \leq s_1 < s_2 < \cdots < s_{n+1}$. There must be two numbers in S that are consecutive, because otherwise $s_2 \geq s_1 + 2 \geq 3$, $s_3 \geq s_2 + 2 \geq 5$, $\ldots$, $s_{n+1} \geq s_n + 2 \geq 2n + 1$, a contradiction. Thus, for some k, S contains k and $k + 1$. By Proposition 6.13, $\gcd(k + 1, k) = \gcd(1, k) = 1$.

10.7 *Numbers fell off the face of a clock, and were replaced in random order.* Label the positions of the clock as $1, 2, \ldots, 12$ (in the usual order). Let a_i be the number which is placed at position i. Let s_i be the sum of the three numbers starting at position i and going "clockwise" (e.g. $s_1 = a_1 + a_2 + a_3$ and $s_{11} = a_{11} + a_{12} + a_1$). Every number belongs to exactly three such sums, so

$$\sum_{i=1}^{12} s_i = 3\sum_{i=1}^{12} i = 3\frac{(12)(13)}{2} = 234.$$

The average of the sums is 19.5, so at least one of them must be ≥ 20. Similarly, the average of the sum of five consecutive numbers is $5\frac{12 \cdot 13}{2} = 390$. The average sum is 32.5, so at least one sum is ≥ 33.

10.9 *Pigeonhole generalization.* The minimum number of objects required is $p_1 + \cdots + p_k - k + 1$. Then at least one class i contains at least p_i objects. If every class contains $\leq p_i - 1$ objects, the total number of objects would be $\leq p_1 + p_2 + \cdots + p_k - k$.

10.14 *Three mutual friends or not friends.* Call the students $1, 2, 3, 4, 5, 6$. By the pigeonhole principle, either student #1 has 3 friends or three who are not friends. Say there are three (or more) friends. If any two of these know each other, then these two and student #1 all know each other. Otherwise, none of them know each other. If there are 3 who are not friends with student #1, either two of them don't know each other, in which case they and student #1 mutually don't know each other, or they all know each other.

10.17 Take the list $n, n-1, \ldots, 1, 2n, 2n-1, \ldots, n+1, 3n, 3n-1, \ldots, 2n+1, \ldots, n^2, n^2-1, \ldots, n^2-n+1$ (that is, n blocks of n numbers, each decreasing).

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HW# 11

10.25 *Decimal n-tuples containing 1,2,3.* We use inclusion-exclusion. The number is

$$10^n - T_1 - T_2 - T_3 + T_{12} + T_{13} + T_{23} - T_{123},$$

where $T_{a_1 \cdots a_k}$ is the number of n-tuples that are missing the digits $a_1, \ldots, a_k$. We have $T_{a_1} = 9^n$ for single a_1, $T_{a_1 a_2} = 8^n$ for any a_1, a_2, and $T_{123} = 7^n$. Thus, the total number of n-tuples is

$$10^n - 3 \cdot 9^n + 3 \cdot 8^n - 7^n.$$

10.29 *No divisors in* $\{6, 10, 15\}$. Let A_1, A_2 and A_3 be the multiples of 6, 10, and 15, respectively, up to 20000. By inclusion-exclusion, the number of integers in question is

$$20000 - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|.$$

We have $|A_1| = \lfloor \frac{20000}{6} \rfloor$, $|A_2| = \lfloor \frac{20000}{10} \rfloor$ and $|A_3| = \lfloor \frac{20000}{15} \rfloor$. The multiples of both 6 and 10 are precisely the multiples of 30, which is the least common multiple of 6 and 10. Thus $|A_1 \cap A_2| = \lfloor \frac{20000}{30} \rfloor$. Similarly $|A_1 \cap A_3| = \lfloor \frac{20000}{30} \rfloor$ and $|A_2 \cap A_3| = \lfloor \frac{20000}{30} \rfloor$. Finally, $|A_1 \cap A_2 \cap A_3| = \lfloor \frac{20000}{30} \rfloor$, since 30 is the smallest number that is a multiple of 6, 10 and 15. The final answer is

$$20000 - \left\lfloor \frac{20000}{6} \right\rfloor - \left\lfloor \frac{20000}{10} \right\rfloor - \left\lfloor \frac{20000}{15} \right\rfloor + 3 \left\lfloor \frac{20000}{30} \right\rfloor - \left\lfloor \frac{20000}{30} \right\rfloor = 14666.$$

10.32 **BONUS.** *Permutations of* $[n]$ *with no odd number as a fixed point.* Let A_i be the set of permutations for which the number $2i - 1$ is a fixed point. What we want to count is the number, E_n, of permutations which are not in any of A_1, $A_2, \ldots, A_k$, where $k = \lfloor (n+1)/2 \rfloor$. By inclusion-exclusion,

$$E_n = \sum_{S \subseteq \{1, \ldots, k\}} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|.$$

For each S, $\cap_{i \in S} A_i$ is the set of permutations with $|S|$ fixed positions, so the size of this set is $(n - |S|)!$. There are $\binom{k}{j}$ sets S of size j, so that

$$E_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (n - j)!.$$

13.8 *True or False: If S is a bounded set of real numbers, and S contains* $\sup S$ *and* $\inf S$*, then S is a closed interval.* This is false. For example, take $S = \{1, 2\}$. S is not an interval, but is bounded and contains $\sup S$ and $\inf S$.

13.10 *Every positive irrational number is the limit of a nondecreasing sequence of rational numbers.* TRUE. Let $x > 0$, $x \notin \mathbb{Q}$. Let $a_1 = 0$. Once we've constructed $a_1, \cdots, a_{n-1}$, her's how to construct a_n. Let $b = \max(a_n, x - 1/n)$. Then $b < x$, so there is at least one rational number in (b, x). Let a_n be one such rational number. Since $b \geq a_n$, we have $a_n > a_{n-1}$. Thus, the sequence (a_n) is increasing. Also, $x - 1/n < a_n < x$ for all n. Given $\epsilon > 0$, if $n > 1/\epsilon$ then $x - \epsilon < a_n < x$. This proves that $a_n \to x$.

13.11 *True or False: Suppose* $\langle a \rangle$ *and* $\langle b \rangle$ *converge. (a) If* $\lim a_n < \lim b_n$*, there exists* $N \in \mathbb{N}$ *such that* $n \geq N$ *implies* $a_n < b_n$*.* This is true. Let $A = \lim a_n$ and $B = \lim b_n$. Let $\epsilon = \frac{1}{3}(B - A)$. By the definition of limits, for some $N_1 \in \mathbb{N}$, $n \geq N_1$ implies $|a_n - A| < \epsilon$ and for some $N_2 \in \mathbb{N}$, $n \geq N_2$ implies $|b_n - B| < \epsilon$. Therefore, for $n \geq \max(N_1, N_2)$,

$$a_n < A + \epsilon < B - \epsilon < b_n.$$

(b) If $\lim a_n \leq \lim b_n$*, there exists* $N \in \mathbb{N}$ *such that* $n \geq N$ *implies* $a_n \leq b_n$*.* This is false. For example, let $a_n = 1/n$ and $b_n = -1/n$. Both $\langle a \rangle$ and $\langle b \rangle$ converge to zero, but $a_n > b_n$ for every n.

13.22 *For each set* S *below, determine whether* S *is bounded, and determine* $\sup S$ *and* $\inf S$*, if they exist.* After moving all terms to one side of the inequalities and factoring, we obtain

$$\begin{array}{ll} \text{(a)} & S = \{x : x(x-5) < 0\} = (0, 5), \\ \text{(b)} & S = \{x : x(x-1)^2 > 0\} = \{x : x > 0 \text{ and } x \neq 1\}, \\ \text{(c)} & S = \{x : x(x^2 - 4x + 1) < 0\} = \{x : x < 0 \text{ or } 2 - \sqrt{3} < x < 2 + \sqrt{3}\}. \end{array}$$

For (a), S is bounded, $\sup S = 5$, $\inf S = 0$. For (b), S is not bounded, $\sup S$ does not exist, $\inf S = 0$. For (c), S is not bounded, $\sup S = 2 + \sqrt{3}$, $\inf S$ does not exist.

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HW# 12

13.20 *Give a sequence in S converging to* $\sup S$ *and a sequence in S converging to* $\inf S$. (a) $S = [0, 1)$. Here $\sup S = 1$ and one can take for example $a_n = 1 - 1/n$ as a sequence with limit 1, and all numbers in S. Next, $\inf S = 0$ and one can take, for example, $a_n = 1/(n+1)$.

(b) $S = \{\frac{2+(-1)^n}{n} : n \in \mathbb{N}\}$. When $n = 2$, $\frac{2+(-1)^n}{n} = 3/2$. For $n = 1$, $\frac{2+(-1)^n}{n} = 1$ and for $n \geq 3$, $\frac{2+(-1)^n}{n} \leq 3/n \leq 1$. Thus, $\sup S = 3/2 = \max S$. We can take the sequence $a_n = 3/2$ for all $n \in \mathbb{N}$ as a sequence of elements in S converging to $\sup S$. Next, $\inf S = 0$, because all elements of S are positive and the sequence $a_m = \frac{1}{2m+1}$ is a sequence in S converging to zero.

13.23 *A, B have upper bounds, $C = \{a+b : a \in A, b \in B\}$. Prove C has an upper bound and* $\sup C = \sup A + \sup B$. Since A and B have upper bounds, $\sup A$ and $\sup B$ exist. Suppose $\sup A = \alpha$ and $\sup B = \beta$. For all $a \in A$ and $b \in B$, $a \leq \alpha$ and $b \leq \beta$, so $a + b \leq \alpha + \beta$. Thus $\alpha + \beta$ is an upper bound for C. By Proposition 13.15, there is a sequence $\langle a \rangle$ of numbers in A for which $a_n \to \alpha$, and there is a sequence $\langle b \rangle$ of numbers in B for which $b_n \to \beta$. Suppose $\sup C = c < \alpha + \beta$. Take $\epsilon = \frac{1}{2}(\alpha + \beta - c)$. Since $a_n \to \alpha$, for some $N_1 \in \mathbb{N}$, $n \geq N_1$ implies $|a_n - \alpha| < \epsilon$. Since $b_n \to \beta$, for some $N_2 \in \mathbb{N}$, $n \geq N_2$ implies $|b_n - \beta| < \epsilon$. Thus, for $n \geq \max(N_1, N_2)$,

$$a_n + b_n > (\alpha - \epsilon) + (\beta - \epsilon) = \alpha + \beta - 2\epsilon = c,$$

which contradicts the assumption that c is an upper bound for C. Thus, $\sup C = \alpha + \beta$.

13.25 *Prove* $\sqrt{1 + n^{-1}} \to 1$. Since $(1 + 1/n)^2 = 1 + 2/n + 1/n^2 > 1 + 1/n$, taking square roots gives $1 + 1/n > \sqrt{1 + 1/n}$. Let $\epsilon > 0$ and suppose that $N > 1/\epsilon$ with $N \in \mathbb{N}$. Then, for $n \geq N$,

$$1 - \epsilon < 1 < \sqrt{1 + 1/n} < 1 + 1/n \leq 1 + 1/N < 1 + \epsilon.$$

This implies that $|\sqrt{1 + 1/n} - 1| < \epsilon$ for $n \geq N$. Therefore, $\sqrt{1 + n^{-1}} \to 1$.

13.26 *If* $\lim a_n = 1$, *show* $\lim(1 + a_n)^{-1} = 1/2$. Let $0 < \epsilon < 1/2$. Since $\lim a_n = 1$, for some $N \in \mathbb{N}$, if $n \geq N$ then $|a_n - 1| < \epsilon$. Thus, for $n \geq N$,

$$\left|\frac{1}{1 + a_n} - \frac{1}{2}\right| = \left|\frac{1 - a_n}{2 + 2a_n}\right| \leq \frac{\epsilon}{4 - 2\epsilon} < \epsilon.$$

Therefore, $\lim(1 + a_n)^{-1} = \frac{1}{2}$.

13.29 *Let $x_n = \frac{1+n}{1+2n}$. Prove* $\lim x_n$ *exists using Monotone Convergence. Prove $x_n \to 1/2$ using the definition of a limit.* For $n \geq 1$, $x_n > \frac{n+1/2}{2n+1} = \frac{1}{2}$. Thus, $\langle x \rangle$ has a lower bound. Next, for all $n \in \mathbb{N}$,

$$x_{n+1} - x_n = \frac{2 + n}{3 + 2n} - \frac{1 + n}{1 + 2n} = \frac{-1}{(3 + 2n)(1 + 2n)} < 0,$$

so $\langle x \rangle$ is non-increasing. By the Monotone Convergence Theorem $\langle x \rangle$ has a limit. To find the limit, rewrite $x_n = 1/2 + \frac{1}{2+4n}$. For any $\epsilon > 0$, let N be an integer satisfying $N > 1/\epsilon$. For $n \geq N$,

$$\frac{1}{2} - \epsilon < \frac{1}{2} < x_n \leq \frac{1}{2} + \frac{1}{4N + 2} < \frac{1}{2} + \frac{1}{4N} < \frac{1}{2} + \epsilon.$$

Therefore, by the definition of a limit, $x_n \to 1/2$.

13.30 *Limit of the sequence* $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Each term in the sum for x_n is $\leq \frac{1}{n+1}$ and there are n summands, so $x_n \leq \frac{n}{n+1} < 1$ for all n. Thus, $\langle x \rangle$ is bounded above. Next we look at the difference $x_{n+1} - x_n$, which equals

$$\frac{1}{2n + 1} + \frac{1}{2n + 2} - \frac{1}{n + 1} > \frac{1}{2n + 2} + \frac{1}{2n + 2} - \frac{1}{n + 1} = 0.$$

Thus, $\langle x \rangle$ is nondecreasing. By the Monotone Convergence Theorem, $\langle x \rangle$ has a limit.

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HW# 13

14.3 *Examples where* $\lim a_n = 0$, $\lim b_n$ *does not exist, and one other condition.* (a) $\lim(a_n b_n) = 0$. Let $a_n = 0$ and $b_n = n$. (b) $\lim a_n b_n = 1$. Take for example $a_n = 1/n$ and $b_n = n$. (c) $\lim(a_n b_n)$ does not exist. Take for example $a_n = 1/n$ and $b_n = n^2$.

14.8 *Statements about a sequence* $\langle x\rangle$. (a) "If $\langle x\rangle$ is unbounded, then $\langle x\rangle$ has no limit." TRUE. Suppose $x_n \to L$, let $\epsilon = 1$ and let N be any natural number. Since $\langle x\rangle$ is unbounded, there is number x_n with $n \geq N$ such that $|x_n| > |L| + 1$. Thus the statement "for $n \geq N$, $|x_n - L| < 1$" is false for every choice of N. This contradicts $x_n \to L$, hence $\langle x\rangle$ has no limit. (b) "If $\langle x\rangle$ is not monotone, then $\langle x\rangle$ has no limit." FALSE. For example, take $x_n = (-1)^n/n$. Then $x_n \to 0$ but $\langle x\rangle$ is not monotone.

14.12 *True or False: If* $a_n \to 0$ *and* $b_n \to 0$, *then* $\sum a_n b_n$ *converges.* FALSE. For example, if $a_n = b_n = 1/\sqrt{n}$ then $a_n \to 0$, $b_n \to 0$ and $\sum a_n b_n = \sum 1/n$ diverges.

14.13 *If* $\langle a\rangle$ *converges, then every subsequence of* $\langle a\rangle$ *converges and has the same limit as* $\langle a\rangle$. Suppose $a_n \to L$ and let $\langle b\rangle$ be a subsequence of $\langle a\rangle$ with $b_k = a_{n_k}$. For any $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $n \geq N$ implies $|a_n - L| < \epsilon$. Let K be the minimum k so that $n_k \geq N$. Then, for $k \geq K$, $|b_k - L| = |a_{n_k} - L| < \epsilon$. Thus, $\langle b\rangle$ satisfies the definition of convergence to L.

14.15 *Suppose* $b \leq L + \epsilon$ *for all* $\epsilon > 0$. *Prove that* $b \leq L$. Assume $b > L$. Then $b = L + c$ for some positive c. Taking $\epsilon = c/2$ gives $b \leq L + c/2$, a contradiction.

14.23 *If* $f_1(x) = x$ *for* $x \in \mathbb{R}$ *and* $f_{n+1}(x) = (f_n(x))^2/2$ *for* $n \geq 1$, *analyze the sequence* $f_1(x), f_2(x), \ldots$ *for various* x. If the sequence has a limit L, then $L = L^2/2$, which implies that either $L = 0$ or that $L = 2$. If $c = 0$ or $c = 2$, then $c^2/2 = c$, and it follows that the sequence is constant if $x = 0$ or $x = 2$. If $0 < c < 2$, then $0 < c^2/2 < c < 2$. Thus, if $0 < x < 2$, the sequence is decreasing and bounded below by 0. By the Monotone Convergence Theorem, the sequence converges. As every term is $\leq c$, the limit is $\leq c$. By the earlier remark, the limit must be 0. Similarly, if $-2 < x < 0$, then $f_2(x) = x^2/2 \in (0, 2)$, and thereafter the sequence decreases to a limit 0. If $c > 2$, then $c^2/2 > c$. Thus, if $x > 2$ then the sequence is increasing. If it has a limit L, then $L \geq c$ since c is the smallest term. The only possible limits are 0 and 2, so the sequence diverges. Similarly, if $x < -2$, then $x^2/2 > 2$, and thereafter the sequence increases with no limit.

14.33 *Suppose* $\sum a_k = A$ *and* $\sum b_k = B$. *Prove that* $\sum(a_k + b_k) = A + B$. Let $s_n = a_1 + \cdots + a_n$ and $t_n = b_1 + \cdots + b_n$. Since $s_n \to A$ and $t_n \to B$, $(s_n + t_n) \to A + B$ by Theorem 14.5 part (b). Since $s_n + t_n$ is the nth partial sum of the series $\sum_{k=1}^{\infty}(a_k + b_k)$, $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

14.55 (**BONUS**). *True or False: If* $a_k \to 0$ *and the sequence of partial sums of the series* $\sum a_k$ *are bounded, then the series converges.* FALSE. For example, consider the sequence $\langle a\rangle$ defined as follows: Let $a_1 = 1$. Every $n \geq 2$ satisfies $2^k < n \leq 2^{k+1}$ for some integer $k \geq 0$. Let $a_n = (-1)^{k+1}2^{1-k}$ for such n. The sequence begins $1, -2, 1, 1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, \ldots$. Since $k \to \infty$ as $n \to \infty$, $a_k \to 0$. Let $s_n = a_1 + \cdots + a_n$. Using an induction argument, it is easy to prove that from $n = 2^k$ to 2^{k+1}, s_n moves from $(-1)^k$ to $(-1)^{k+1}$ (that is, from 1 to -1 or vice versa) in increments of 2^{-k}. Thus, $|s_n| \leq 1$ for all n, and s_n oscillates between 1 and -1 infinitely often, so does not have a limit.