Math 453 – Number Theory

Practice Final Exam (from Fall 2005)

1. (20 points)

   (a) Find $\tau(2100)$, $\phi(2100)$ and $\sigma(72)$.

   (b) Find $(2100, 72)$ and $[2100, 72]$.

   (c) Find the last two base-10 digits of 17^{2041}.

   (d) Find the value of the Legendre symbol $(\frac{73}{107})$.

2. (12 points) Suppose $(a, b) = d$ and $d \nmid g$. Prove that the equation $ax^2 + by^3 = g$ has no solutions with integers x, y.

3. (12 points) Solve the system of congruences

$$x \equiv 2 \pmod{7}$$
$$x \equiv 1 \pmod{10}$$
$$x \equiv 3 \pmod{11}$$

4. (12 points) Suppose p is an odd prime and a is a quadratic residue of p. Prove that a is not a primitive root of p.

5. (12 points) Prove that if p is prime, $p > 5$ and $p \equiv 1 \pmod{4}$, then the congruence $x^2 \equiv -1 \pmod{5p}$ has a solution.

6. (18 points)

   (a) Show that 2 is a primitive root of 37.

   (b) Use part (a) to find a number b with $\text{ord}_{37} b = 9$.

   (c) Use part (a) to find two other primitive roots of 37 that are between 1 and 37.

7. (14 points) Suppose q is prime, $n \geq 2$ and $a^{q^2} \equiv 1 \pmod{n}$. Show that $\text{ord}_n a = q^2$ if and only if $a^q \not\equiv 1 \pmod{n}$.

BONUS A. (15 points) Find all numbers m with $\lambda(m) = 12$. There are more than 100 such numbers, so you need to describe the numbers in terms of prime factorizations. A complete proof is required.

BONUS B. (15 points) Prove that $\sigma(n)\phi(n) \leq n^2$ for all positive integers n.