Some sieve estimates

Definitions

\[ \Phi(x, z) = \left| \{ n \leq x : p^{-}(n) > z \} \right| \]

Note: 1 is always counted in \( \Phi(x, z) \), since we consider \( p^{-}(1) = \infty \)

\[ \Psi(x, y) = \left| \{ n \leq x : p^{+}(n) \leq y \} \right| = \text{number of } y\text{-smooth integers } \leq x \]

Again, 1 is always counted.

Heuristic for \( \Phi(x, z) \)

\[ \Phi(x, z) = x \cdot P\left[p^{-} n \text{ for } p = z\right] \approx x \prod_{p \leq z} \left(1 - \frac{1}{p}\right) \approx \frac{x e^{-r}}{\log z} \]

This is correct for \( z = x^{o(1)} \) (sieve methods), but a little off when \( z \) is large. For example, if \( z = \sqrt{x} \), then

\[ \Phi(x, z) = \pi(x) - \pi(\sqrt{x}) + 1 \approx \frac{x}{\log x} , \]

whereas the above heuristic gives \( \Phi(x, z) \approx 2e^{-r} \frac{x}{\log x} \).

Theorem 06 Uniformly for \( x \geq x^{2} z \geq 4 \), we have

\[ \Phi(x, z) \asymp \frac{x}{\log z} \]

Moreover, the upper bound holds for \( x \geq z \geq 2 \).

Recall \( f \leq g \) means \( \exists \) pos. constants \( c_{1}, c_{2} \) so that \( c_{1}g \leq f \leq c_{2}g \),

some \( \ll f \ll g \) and \( f \ll g \).

Proof

Thm 06 upper bound is an exercise, using Theorem 01.

Thm 06 lower bound: when \( z = x^{100} \), we have

\[ \Phi(x, z) \geq \Phi(x, \frac{x}{z}) = \pi(x) - \pi(\frac{x}{z}) + 1 \]

\[ \gg \frac{x}{\log x} \geq \frac{x}{100 \log z} \cdot \]

when \( z = x^{100} \), the bound is a standard application of sieve methods. We give here an alternative approach.
First, let \( x_0 \) be some large constant. If \( x < x_0 \), then
\[
\Phi(x, z) \geq \sum_{n \leq x} \frac{1}{\log n} \geq \sum_{p \leq x} \frac{1}{\log p} \quad n = pm
\]
\[
= \sum_{z^p \leq x} \frac{1}{\log p} \sum_{m \leq x/z} \frac{1}{\log p} = \sum_{z < p \leq x/z} \sum_{m \leq x/z} \frac{1}{\log p}
\]
\[
> \sum_{m \leq x^{1/2}} \left( \Theta \left( \frac{x}{m} \right) - \Theta(z) \right)
\]

Since \( x > x_0 \) and \( z \leq x^{100} \), \( \Theta \left( \frac{x}{m} \right) - \Theta(z) \gg \frac{x}{m} \) by Chebyshev's estimates.

Thus
\[
(1) \quad \Phi(x, z) \gg x \sum_{m \leq x^{1/2}} \frac{1}{m} \geq \frac{x}{\log x} \sum_{m \leq x^{1/2}} \frac{x(m)}{m}
\]

where \( x \) is multiplicative, \( x(p) = 1 \) if \( z < p < x^{1/4} \) and \( x(p^r) = 0 \) otherwise.

Then
\[
\sum_{m \leq x^{1/2}} \frac{x(m)}{m} = \sum_{m = 1}^{\infty} \frac{x(m)}{m} - \sum_{m > x^{1/2}} \frac{x(m)}{m}
\]
\[
> \prod_{z < p \leq x^{1/4}} \left( 1 + \frac{1}{p} \right) - \sum_{m > x^{1/2}} \frac{x(m)}{m} \frac{\log m}{\log x}
\]
\[
> \prod_{z < p \leq x^{1/4}} \left( 1 + \frac{1}{p} \right) - \frac{2}{\log x} \sum_{m = 1}^{\infty} \frac{x(m) \log m}{m}
\]

Let \( f(s) = \prod_{z < p \leq x^{1/4}} \left( 1 + p^{-s} \right) = \sum_{m = 1}^{\infty} \frac{x(m)}{m^s} \). We have
\[
\sum_{m = 1}^{\infty} \frac{x(m) \log m}{m^s} = - f'(s) = f(s) \left( 1 - \frac{f(s)}{f'(s)} \right) = f(s) \sum_{z < p \leq x^{1/4}} \frac{\log p}{p^{s+1}}
\]

At \( s = 1 \), we have \( \sum_{z < p \leq x^{1/4}} \frac{\log p}{p} \leq \sum_{p \leq x^{1/4}} \frac{\log p}{p} = \frac{1}{4} \log x + O(1) \). Hence
\[
\sum_{m \leq x^{1/4}} \frac{x(m)}{m} \approx f(1) \left( 1 + O \left( \frac{1}{\log x} \right) \right) \gg f(1) = \prod_{z < p \leq x^{1/4}} \left( 1 - \frac{1}{p} \right) \prod_{p \leq x^{1/4}} \frac{1}{p-1}
\]
\[
\geq \prod_{p \leq x^{1/4}} \left( 1 - \frac{1}{p} \right) = \prod_{p \leq x^{1/4}} \frac{1}{p-1} \gg \frac{\log x}{\log z}.
\]
By (1) and (2), \( \Phi(x, z) \geq \frac{x}{\log z} \). 

**Theorem 05** Uniformly for \( x \geq y \geq 2 \), we have 

\[
\Psi(x, y) \ll x e^{-\frac{1}{2} \frac{\log x}{\log y}}.
\]

**Proof.** First case, \( y \leq e^4 \). Let \( P_1, P_2, \ldots, P_{16} \) be the primes \( \leq e^4 \). Then 

\[
P^+(n) \leq y \Rightarrow n = p_1^{e_1} \cdots p_{16}^{e_{16}} \text{ with } e_j \leq \frac{\log x}{\log P_j} \text{ for each } j.
\]

Hence 

\[
\Psi(x, y) \leq \frac{1}{16} \sum_{j=1}^{16} \left( \frac{\log x}{\log P_j} \right) \ll \left( \log x \right)^{16} \ll x^{\frac{1}{16}} x \frac{1}{2} \frac{\log x}{\log y} \ll x e^{-\frac{1}{2} \frac{\log x}{\log y}}.
\]

Now suppose \( y > e^4 \), and set \( \delta = \frac{4}{\log y} \leq \frac{1}{2} \). Then 

\[
\Psi(x, y) \leq x^{\frac{1}{16}} + \sum_{x^{\frac{1}{16}} < n \leq x \atop P^+(n) \leq y} 1
\]

\[
\leq x^{\frac{1}{16}} + \sum_{n \leq x \atop \delta \leq \frac{n^{1/16}}{x^{1/16}}} \delta = x^{\frac{1}{8}} + x^{-\frac{1}{2}} \delta \sum_{n \leq x} f(n),
\]

where \( f \) is multiplicative and 

\[
f(p^k) = \begin{cases} 
p^{\delta k} & p \leq y \\
1 & p > y \end{cases}.
\]

Since \( p^{\delta k} = (e^{4\delta})^k < 1.8^k \), the conditions of Theorem 01 hold. 

Thus 

\[
\sum_{n \leq x} f(n) \ll \frac{x}{\log x} \prod_{p \leq y} \left( 1 + p^{\delta - 1} + p^{2(\delta - 1)} + \ldots \right)
\]

\[
\leq \frac{x}{\log x} \exp \left\{ \sum_{p \leq y} \left( p^{\delta - 1} + O\left( p^{2(\delta - 1)} \right) \right) \right\}
\]

\[
\ll \frac{x}{\log x} \exp \left\{ \sum_{p \leq y} \left( \frac{1}{p} + \frac{p^{\delta - 1}}{p} \right) \right\}
\]

Since \( \delta \log p \leq \delta \log y \leq \frac{y}{4} \), by the Mean Value Theorem, 

\[
p^{\delta - 1} = e^{\delta \log p - 1} = \delta \log p \quad \text{for } 0 \leq \theta \leq \delta \log p \leq \frac{y}{4},
\]

so 

\[
p^{\delta - 1} \ll 2 \delta \log p.
\]
Hence
\[
\sum_{x \leq k} f(x) \ll \frac{x}{\log x} \exp \left\{ \log \log x + o(1) + 2\delta \sum_{p \leq y} \frac{\log p}{p} \right\} \cdot \frac{\log x}{\log y} \exp \left\{ 2\delta (\log y + o(1)) \right\} \ll x.
\]

In conclusion,
\[
\psi(x,y) = x^{\frac{1}{2}} + x^{1 - \frac{1}{2}\delta} = x^{\frac{1}{2}} + xe^{-\frac{1}{2} \frac{\log x}{\log y}} \ll xe^{-\frac{1}{2} \frac{\log x}{\log y}}.
\]

Remarks If y is not too small, in fact \(\psi(x,y) \sim g(u) x\), \(u = \frac{\log x}{\log y}\),
\(g(u) = u - (u + o(u))\).

**Theorem 07** Uniformly for \(z \leq y \leq z \leq x\),
\[
\left| \sum_{p \leq y} \frac{x}{\log x} \right| \ll xe^{-\frac{1}{2} \frac{\log x}{\log y}}.
\]

**Proof** let \(a = \prod_{p \leq y} p^x\) and \(n = ab\), so \(p(a) \leq y < p(b)\).

If \(z\) is large compared to \(y\), Theorem 05 says that \(a \gg z\)
should be rare. The number of \(n\) with \(b = 1\) is \(\psi(x,y)\).
If \(b > 1\), then \(b > y\) and \(a < \frac{x}{y}\). Given \(a\), the number
of \(b\) is \(\ll \psi(x,y) \ll \frac{x}{\log y}\).

by **Theorem 06**, Thus, the number of \(n\) in question is
\[
\ll xe^{-\frac{1}{2} \frac{\log x}{\log y}} + \frac{x}{\log y} \sum_{\substack{p(a) \leq y \leq z \leq x \leq b \leq X}} \frac{1}{a}.
\]

By partial summation, the sum on \(a\) is
\[
\psi(x,y) - \psi(z,y) + \int_{z}^{x} \frac{\psi(t,y)}{t} dt
\]
\[
\ll e^{-\frac{1}{2} \frac{\log x}{\log y}} + \int_{z}^{x} \frac{1}{t} e^{-\frac{1}{2} \frac{\log t}{\log y}} dt = e^{-\frac{1}{2} \frac{\log x}{\log y}} + \log x e^{-\frac{1}{2} \frac{\log x}{\log y}} \ll (\log y) e^{-\frac{1}{2} \frac{\log x}{\log y}}.
\]