Problem 1
Let \( f(n) = \phi(n)/n \), and let \( \{n_k\}_{k=1}^\infty \) be the sequence of values \( n \) at which \( f \) attains a “record low”; i.e., \( n_1 = 1 \) and, for \( k \geq 2 \), \( n_k \) is defined as the smallest integer > \( n_{k-1} \) with \( f(n_k) < f(n) \) for all \( n < n_k \). (For example, since the first few values of the sequence \( f(n) \) are \( 1, 1/2, 2/3, 1/2, 4/5, 1/3, \ldots \), we have \( n_1 = 1, n_2 = 2, \) and \( n_3 = 6, \) and the corresponding values of \( f \) at these arguments are \( 1, 1/2, \) and \( 1/3 \).) Find (with proof) a general formula for \( n_k \) and \( f(n_k) \).

Problem 2
Find all arithmetic functions \( f \) satisfying the given relation:

(i) \( f * f = e \)
(ii) \( f * f = f \)
(iii) \( f * f = 1 \)

Problem 3
Find a “simple” evaluation of each function:

(i) \( g_k(n) = \sum_{d|n, (d, k) = 1} \mu(d) \), where \( k \in \mathbb{N} \) is fixed.
(ii) \( f = \mu * \mu^2 \).

Problem 4
Assume \( f \) is multiplicative. Prove that:

(a) \( f^{-1}(n) = \mu(n)f(n) \) for square-free \( n \).
(b) \( f^{-1}(p^a) = f(p)^2 - f(p^2) \) for every prime \( p \).
(c) Show that if \( f \neq e \) and \( f \) is completely multiplicative, then \( f^{-1} \) is not completely multiplicative.

Problem 5
Suppose \( f \) is multiplicative.

(a) Prove that \( \sum_{d|n} f(d) = \prod_{p^a|n} (1 + f(p) + f(p^2) + \cdots + f(p^a)) \).
(b) Suppose \( \sum_{n=1}^\infty |f(n)| \) converges. Show that \( \sum_{n=1}^\infty f(n) = \prod_p (1 + f(p) + f(p^2) + \cdots) \).

Problem 6
Let \( f(n) = |\{(n_1, n_2) \in \mathbb{N}^2 : [n_1, n_2] = n\}| \), where \([n_1, n_2]\) is the least common multiple of \( n_1 \) and \( n_2 \). Show that \( f \) is multiplicative and evaluate \( f \) at prime powers.

Problem 7
An arithmetic function \( f \) is called periodic if there exists a positive integer \( k \) such that \( f(n+k) = f(n) \) for all \( n \in \mathbb{N} \); the integer \( k \) is called a period for \( f \). Show that if \( f \) is completely multiplicative and periodic, then the values of \( f \) are either zero or roots of unity. (A root of unity is a complex solution of \( z^n = 1 \) for some \( n \in \mathbb{N} \).)