Problem 1
Let \( M(x) = \sum_{n \leq x} \mu(n) \). Using Perron inversion with \( 1/\zeta(s) \), show that
\[
M(x) = O(xe^{-c\sqrt{\log x}}),
\]
where \( c \) is a positive constant. You may use without proof the following fact: For some positive \( B \),
\[
\left| \frac{1}{\zeta(s)} \right| \ll \log(|t| + 2) \quad \left( \sigma \geq 1 - \frac{B}{\log(|t| + 2)}, \text{ all } t \right).
\]

Problem 2
Prove the following properties of the \( \Gamma \)-function.
(a) for all real \( x \) not equal to a nonpositive integer and all \( y \neq 0 \), prove that \( |\Gamma(x+iy)| < |\Gamma(x)| \).
(b) Show that \( \frac{\Gamma'(1/2)}{\Gamma(1/2)} = -\gamma - 2 \log 2 \).
(c) Show that \( \Gamma(s) \) is increasing for real \( s \geq \frac{3}{2} \).

Problem 3
(a) Make a table of the Dirichlet characters modulo 24 (a table listing each character and its value at the numbers coprime to 24).
(b) Make a multiplication table for the character group \( C_{24} \).

Problem 4
The claim in this problem may seem surprising at first, but it’s easy to prove!
Show that if every arithmetic progression \( l \mod k \) with \( (l,k) = 1 \) contains at least one prime, then every such progression contains infinitely many primes.

Problem 5
**Bonus Problem (20 points).** Let \( f \) be an arithmetic function with \( f(1) \neq 0 \), and \( g \) the convolution inverse of \( f \), and let \( F(s) \) and \( G(s) \) be the Dirichlet series associated with \( f \) and \( g \). In general, the convergence of \( F(s) \) in some half-plane does not imply that \( G(s) \) also converges in the same half-plane. In class, an example was given \((f(1) = 1, f(2) = -1, \text{ and } f(n) = 0 \text{ for } n \geq 2)\) for which \( F(s) \) was convergent for all \( s \), but \( G(s) \) was convergent only in the half-plane \( \sigma > 0 \). Changing \( f(2) \) to \( f(2) = -2^\alpha \), where \( \alpha \in \mathbb{R} \), \( G(s) \) converges in the half-plane \( \sigma > \alpha \). Is it possible that \( F(s) \) converges everywhere, but \( G(s) \) converges nowhere?