Problem 1
Given positive integers \( l \) and \( k \) with \((l, k) = 1\), express the Dirichlet series \( \sum_{n \equiv l \pmod{k}} \mu(n)n^{-s} \) in terms of Dirichlet \( L \)-functions.

Problem 2
Let \( \chi \) be a non-principal character modulo \( k \), and let \( t \neq 0 \).
(a) Prove that \(|L^3(\sigma, \chi_0)L^4(\sigma + it, \chi)L(\sigma + 2it, \chi^2)| \geq 1\) for \( \sigma > 1 \), where \( \chi_0 \) is the principal character modulo \( k \).
(b) Use part (a) to show that \( L(1 + it, \chi) \neq 0 \). [Thus, \( L(s, \chi) \neq 0 \) on the line \( \sigma = 1 \).]

Problem 3
Show that if \( f \) is a periodic, completely multiplicative arithmetic function, then \( f \) is a Dirichlet character to some modulus \( k \).

Problem 4
For real \( a \in (0, 1] \), define the Hurwitz zeta function by

\[
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}.
\]

In particular, \( \zeta(s, 1) = \zeta(s) \).
(a) Let \( \chi \) be any Dirichlet character. Express \( L(s, \chi) \) in terms of Hurwitz zeta functions (with varying \( a \)).
(b) Show, for \(-1 < \Re s < 0 \) and real \( 0 < a \leq 1 \), that

\[
\zeta(s, a) = -2^s \pi^{s-1} \Gamma(1 - s) \sum_{n=1}^{\infty} \frac{\sin(\pi s/2 + 2\pi na)}{n^{1-s}}.
\]

Hint: first prove

\[
\zeta(s, a) = -s \int_{-a}^{\infty} \frac{S(x) dx}{(x + a)^{s+1}}\quad (-1 < \sigma < 0), \text{ where } S(x) = \begin{cases} \{x\} - 1/2 & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z}. \end{cases}
\]

(c) Deduce the following functional equation for \( \zeta(s, a) \): For integers \( m, n \) with \( 1 \leq m \leq n \),

\[
\zeta(1 - s, m/n) = \frac{2^s \Gamma(s)}{(2\pi n)^s} \sum_{k=1}^{n} \cos \left( \frac{\pi s}{n} - \frac{2\pi km}{n} \right) \zeta(s, k/n).
\]

[This can be used to prove a functional equation for Dirichlet \( L \)-functions.]

Problem 5
Find a prove a formula for \( f(k) \), the number of \textit{primitive} characters modulo \( k \). Hint: first show that \( f \) is multiplicative.