Math 595SM Homework Problems, Fall 2007

1. (10 points) Let $\mathscr{A} = \{n \le x : \lambda(n) = -1\}$ and let $\mathscr{P}$ be the set of primes $\le \sqrt{x}$. Show that $S(\mathscr{A}, \mathscr{P}) \sim x/\log x$ and, choosing appropriate X and $\rho()$, show that $XV(\mathscr{P}) \sim e^{-\gamma}x/\log x$. (Comparing this with the example given in class with $\mathscr{A} = \{n \le x : \lambda(n) = 1\}$, where we get the same asymptotic for $XV(\mathscr{P})$, we see that sieves cannot distinguish between numbers with an even number of prime factors and numbers with an odd number of prime factors. This phenomenon is known as the "parity problem" of sieve theory).

2. (10 points each) Use theorems from class to prove the following

   (a) Suppose $1 \le a \le b$ and $(a, b) = 1$. Show that the number, N, of primes $p \le x$ for which $ap + b$ is also prime satisfies

   $$N \ll \frac{ab}{\phi(ab)} \frac{x}{\log^2 x}$$

   uniformly in a, b, x, i.e., the constant implied by the $\ll$ symbol is independent of a, b and x.

   (b) Let $\Phi(x, z)$ denote the number of integers $n \le x$ which have no prime factor $\le z$. Prove that uniformly in $2 \le z \le x$, we have

   $$\Phi(x, z) \ll \frac{x}{\log z}.$$

3. (20 points) Show that for every $m \ge 1$ there is a number g so that the following holds. Suppose $F(x) \in \mathbb{Z}[x]$, $F = F_1 \cdots F_m$, where each F_i has positive leading coefficient and is irreducible over $\mathbb{Q}$, no F_i is a rational multiple of any other F_j, and for every prime p, there is an n so that $p \nmid F(n)$. Then, for $x > x_0(F)$, $x_0(F)$ some constant depending on F, we have

   $$|\{n \le x : \forall i, \omega(F_i(n)) \le g\deg(F_i)\}| \gg_F \frac{x}{\log^m x}.$$

4. (10 points) **Almost primes in short intervals.** Prove that for every $\delta > 0$, there is a natural number k so that whenever $x \ge x_0(\delta)$, then the interval $(x, x + x^\delta]$ contains an integer q with $\omega(q) \le k$.

5. As in the proof of Theorem 5', let

$$H_1(x) = \sum_{n \le x} \frac{\mu^2(n)}{\phi(n)}.$$

(a) (15 points) Show that $H_1(x) = \log x + c + o(1)$, where

$$c = \gamma + \sum_p \frac{\log p}{p(p-1)} = 1.332\ldots$$

and $\gamma = 0.5772\ldots$ is Euler's constant.

(b) (25 points) Show that for $x \ge 6$, $H_1(x) \ge \log x + 1$.

6. (15 points) Show the equivalence of the following two forms of the Bombieri-Vinogradov Theorem:

$$\forall A > 0\ \exists B : \sum_{q \le x^{1/2}/\log^B x} \max_{(a,q)=1} \max_{2 \le y \le x} \left| \pi(y;q,a) - \frac{\mathrm{li}(y)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A},$$

and

$$\forall A > 0\ \exists B : \sum_{q \le x^{1/2}/\log^B x} \max_{(a,q)=1} \max_{2 \le y \le x} \left| \psi(y;q,a) - \frac{y}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}.$$

Here

$$\psi(x;q,a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n).$$

7. (20 points) (a) Show, using elementary estimates, that

$$\Psi(x, x^{1/u}) = x(1 - \log u) + O(x/\log x)$$

uniformly for $1 \le u \le 2$. Here $\Psi(x,y)$ is the number of integers $\le x$ whose prime factors are all $\le y$.

(b) Use part (a) and the Polya-Vinogradov inequality to show that

$$n(p) \ll_\varepsilon p^{1/(2\sqrt{e})+\varepsilon}, \qquad \forall \varepsilon > 0.$$

Here $n(p)$ is the least quadratic nonresidue modulo p.